BIR GEOMETRIK TENGSIZLIKNING TURLI ISBOTLARI

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Maqolada 1961-yilda xalqaro matematika olimpiadasida o‘quvchilarga yechish uchun tavsiya etilgan bir geometrik masalaning turli isbotlari keltirilgan.

Kalit so‘zlar: *geometrik tengsizliklar, Koshi tengsizligi, Hadviger-Finsler tengsizligi, nuqtaning affiksi, oriyentirlangan uchburchak yuzi, qabariq funksiya.*

В статье приведены различные доказательства одной геометрической задачи, предложенной для решения участникам международной математической олимпиады в 1961 году .

Ключевые слова: *геометрические неравенства, неравенство Коши, неравенство Хадвигера-Финслера, аффиксы точек, площадь ориентированного треугольника, выпуклая функция.*

This article presents different solution of one of the mathematical tasks on inequalities offered to the participants of the olimpiad of 1961.

Key words: *geometric inequalities, inequality of Koshi, inequality Hadviger-Finslera, affixes of point, area of orientation triangle, convex function.*

1961-yilning 6–16-iyul kunlari Vengriyada bo‘lib o‘tgan xalqaro matematika olimpiadasida quyidagi masala berilgan:

Tomonlarining uzunliklari a, b, c *bo‘lgan uchburchakning yuzi S ga teng bo‘lsin. U holda*

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3} \qquad (1)$$

tengsizlik o‘rinli bo‘lishini isbotlang. Qaysi hollarda tenglik bajariladi?

(1) tengsizlikning turli ko‘rinishdagi mavjud isbotlarining aksariyati [3] adabiyotdan o‘rin olgan. Bu isbotlar bilan birga turli adabiyotlardan qo‘shimcha to‘plangan ba’zi isbotlarni keltiramiz.

1-usul. Ma’lumki, tomoni c ga teng bo‘lgan muntazam uchburchakning balandligi $\frac{c\sqrt{3}}{2}$ ga teng. Endi

tomonlari a,b,c bo‘lgan ixtiyoriy uchburchakning C uchidan shu tomonga o‘tkazilgan CD perpendikulyar uzunligi $\frac{c\sqrt{3}}{2}+y$ bo‘lib, bu tomonni $\frac{c}{2}-x$ va $\frac{c}{2}+x$ kesmalarga ajratsin. U holda ADC va BCD to‘g‘ri burchakli uchburchaklardan Pifagor teoremasiga ko‘ra,

$$a^2+b^2+c^2-4S\sqrt{3}=\left(\frac{c}{2}-x\right)^2+\left(y+\frac{c\sqrt{3}}{2}\right)^2+\left(\frac{c}{2}+x\right)^2+\left(y+\frac{c\sqrt{3}}{2}\right)^2+c^2-4\sqrt{3}\cdot\frac{1}{2}\cdot c\cdot\left(y+\frac{c\sqrt{3}}{2}\right)=$$

$$=2x^2+2\left[\left(y+\frac{c\sqrt{3}}{2}\right)^2-\sqrt{3}c\cdot\left(y+\frac{c\sqrt{3}}{2}\right)+\frac{3c^2}{4}\right]=2x^2+2\left[\left(y+\frac{c\sqrt{3}}{2}-\frac{c\sqrt{3}}{2}\right)\right]^2=2\left(x^2+y^2\right)\geq\square$$

Demak, (1) tengsizlik o‘rinli. Tenglik x=y=0 bo‘lganda, ya’ni ABC muntazam uchburchak bo‘lganda erishiladi.

2-usul. Aniqlik uchun ABC uchburchak tomonlari uzunliklari bo‘yicha $a\leq b\leq c$ shartni qanoatlantirsin. ABC_1 uchburchakni yasaymiz va CC_1 kesma uzunligini hisoblaymiz.

$\angle ABC=\beta$ belgilash kiritsak, $\angle CBC_1=60^0-\beta$ bo‘ladi.

$$AB=c, BC=a,\ CA=b$$

ekanligini e’tiborga olsak,

$$\sin\beta=\frac{2S}{ac},\ \cos\beta=\frac{a^2+c^2-b^2}{2ac}$$

bo‘lib ΔCBC_1 dan kosinuslar teoremasiga ko‘ra

$$CC_1^2=a^2+c^2-2ac\cos\left(60^0-\beta\right)=a^2+c^2-2ac(\cos 60^0\cdot\cos\beta+\sin 60^0\cdot\sin\beta)=$$

$$=a^2+c^2-ac\cos\beta-\sqrt{3}ac\sin\beta=a^2+c^2-ac\cdot\frac{a^2c^2-b^2}{2ac}-\sqrt{3}ac\cdot\frac{2s}{ac}=$$

$$=a^2+c^2-\frac{a^2+c^2-b^2}{2}-2S\sqrt{3}=\frac{1}{2}\left[\left(a^2+b^2+c^2\right)-4S\sqrt{3}\right]$$

$CC_1\geq 0$ bo‘lgani uchun (1) tengsizlik o‘rinli. Tenglikka $\tilde{N}$ nuqta C_1 nuqta bilan ustma-ust tushganda, ya’ni ABC muntazam uchburchak bo‘lganda erishiladi.

3-usul. Teskaridan faraz qilamiz, ya’ni (1) tengsizlik o‘rniga

$$4S\sqrt{3}>a^2+b^2+c^2$$

tengsizlik bajarilsin deb faraz qilamiz. Ammo uchburchakning yuzi uchun

$$S=\frac{1}{2}bc\sin\alpha$$

formuladan foydalanib, bu tengsizlikni

$$2bc\sin\alpha>\frac{1}{\sqrt{3}}\left(a^2+b^2+c^2\right)\qquad(2)$$

ko‘rinishda yozamiz. Kosinuslar teoremasiga ko‘ra,

$$2bc\cos\alpha>b^2+c^2-a^2\qquad(3)$$

(2) va (3) munosabatlarni kvadratga ko‘tarib qo‘shilsa,

$$4b^2c^2 > \frac{1}{3}\left(a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2\right)+a^4+b^4+c^4-2a^2b^2+2b^2c^2-2c^2a^2$$

yoki

$$12b^2c^2 > 4\left(a^4+b^4+c^4\right)+8b^2c^2-4a^2c^2-4a^2b^2$$

bundan esa

$$0 > \left(a^2-b^2\right)^2+\left(b^2-c^2\right)+\left(c^2-a^2\right)^2$$

ziddiyatni hosil qilamiz. Demak, farazimiz noto‘g‘ri. (1) tengsizlik o‘rinli ekan.

4-usul. Uchburchak yuzini hisoblashning

$$S=\sqrt{\frac{a+b+c}{2}\cdot\frac{-a+b+c}{2}\cdot\frac{a-b+c}{2}\cdot\frac{a+b-c}{2}}$$

Geron formalasiga o‘rta arifmetik va o‘rta geometrik miqdorlar orasidagi tengsizlikni qo‘llaymiz.

$$16S^2=(a+b+c)\cdot(-a+b+c)\cdot(a-b+c)\cdot(a+b-c)\le$$
$$\le(a+b+c)\left(\frac{-a+b+c+a-b+c+a+b-c}{3}\right)^3=\frac{(a+b+c)^4}{27}$$

bundan

$$4S\le\frac{(a+b+c)^2}{3\sqrt{3}}=\frac{a^2+b^2+c^2+2ab+2bc+2ca}{3\sqrt{3}}\le\frac{3\left(a^2+b^2+c^2\right)}{3\sqrt{3}}=\frac{a^2+b^2+c^2}{3\sqrt{3}}$$

O‘z navbatida bu tengsizlikdan (1) tengsizlik kelib chiqadi. Tenglik esa $a=b=c$ bo‘lganda erishiladi.

5-усул. $a^2+b^2+c^2$ ifodada a^2 ni kosinuslar teoremasi bo‘yicha $b^2+c^2-2bc\cos\alpha$ ifodaga almashtirib va uchburchak yuzasini yisoblash formulasidan foydalanib quyidagi ko‘rinishda shakl almashtirishlar bajaramiz:

$$a^2+b^2+c^2-4S\sqrt{3}=2\left(b^2+c^2\right)-2bc\cos\alpha-2bc\sqrt{3}\sin\alpha=$$
$$=2\left(b^2+c^2\right)-4bc\left(\frac{1}{2}\cdot\cos\alpha+\frac{\sqrt{3}}{2}\cdot\sin\alpha\right)=2\left(b^2+c^2\right)-4bc\cos\left(60^\circ-\alpha\right)\ge$$
$$\ge 2\left(b^2+c^2\right)-4bc=2\left(b-c\right)^2\ge 0$$

Demak, (1) tengsizlik o‘rinli. Tenglik $b=c$ va $\alpha=60^0$, ya’ni $a=b=c$ bo‘lganda erishiladi.

6-usul. (1) tengsizlikni kvadratga ko‘tarib, shakl almashtirishlar bajaramiz.

$$\left(a^2+b^2+c^2\right)^2\ge 3(a+b+c)(-a+b+c)(a-b+c)(a+b-c),$$

yoki

$$a^4+b^4+c^4+2a^2b^2+2b^2c\ +2c^2a^2\ge 3\left(2a^2b^2+2b^2c\ +2c^2a^2-a^4-b^4-c^4\right),$$

yoki

$$2\left(a^4+b^4+c^4\right)-2\left(a^2b^2+b^2c\ +c^2a^2\right)\geq 0,$$

yoki

$$\left(a^2-b^2\right)^2+\left(b^2-c^2\right)+\left(c^2-a^2\right)^2\geq 0$$

Oxirgi tenglik o‘rinli bo‘lgani uchun, quyidan yuqoriga tomon fikr yuritilsa, (1) tengsizlik ham o‘rinli ekanligi ravshan bo‘ladi.

7-usul. Dastlab uchburchak burchaklarining trigonometrik funksiyalarini ifodalovchi quyidagi munosobatlarni ko‘rsatamiz.

$$tg\frac{A}{2}tg\frac{B}{2}+tg\frac{B}{2}tg\frac{C}{2}+tg\frac{C}{2}tg\frac{A}{2}=1 \qquad (4)$$

$$tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\geq\sqrt{3} \qquad (5)$$

Haqiqatan ham,

$$tg^2\frac{A}{2}=\frac{1-\cos\alpha}{1+\cos\alpha}=\frac{1-\dfrac{b^2+c^2-a^2}{2}}{1+\dfrac{b^2+c^2-a^2}{2}}=\frac{a^2-(b-c)^2}{(b+c)^2-a^2}=\frac{(a-b+c)(a+b-c)}{(a+b+c)(-a+b+c)}=\frac{(p-b)(p-c)}{p(p-a)}$$

bo‘lgani uchun

$$tg\frac{A}{2}tg\frac{B}{2}+tg\frac{B}{2}tg\frac{C}{2}+tg\frac{C}{2}tg\frac{A}{2}\leq tg^2\frac{A}{2}+tg^2\frac{B}{2}+tg^2\frac{C}{2}$$

tengsizlikni hosil qilish qiyin emas.

U holda

$$tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}=\sqrt{\left(tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\right)^2}=$$

$$=\sqrt{tg^2\frac{A}{2}+tg^2\frac{B}{2}+tg^2\frac{C}{2}+2\left(tg\frac{A}{2}tg\frac{B}{2}+tg\frac{B}{2}tg\frac{C}{2}+tg\frac{C}{2}tg\frac{A}{2}\right)}\geq\sqrt{1+2\cdot1}=\sqrt{3}$$

Endi isbotlangan (4) tengsizlikdan foydalanib, (1) tengsizlikni ko‘rsatamiz.

$$a^2+b^2+c^2\geq a^2+b^2+c^2-(a-b)^2-(b-c)^2-(c-a)^2=2ab+2bc+2ca-2\left(a^2+b^2+c^2\right)=$$

$$=2ab+2bc+2ca-2\left(ab\cos C+bc\cos A+ca\cos B\right)=4ab\sin^2\frac{C}{2}+4bc\sin^2\frac{A}{2}+4ca\sin^2\frac{B}{2}$$

Ammo

$$bc=\frac{2S}{\sin A}=\frac{S}{\sin\frac{A}{2}\cdot\cos\frac{A}{2}},\ ca=\frac{2S}{\sin B}=\frac{S}{\sin\frac{B}{2}\cdot\cos\frac{B}{2}},ab=\frac{2S}{\sin C}=\frac{S}{\sin\frac{C}{2}\cdot\cos\frac{C}{2}}$$

bo‘lgani uchun

$$a^2+b^2+c^2\geq 4S\left(tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\right)\geq 4S\sqrt{3}$$

Tenglik $A=B=C=60^0$ bo‘lganda erishiladi.

8-usul. Bu usul ham oldingi usulga o‘xshash bo‘lib, isbotlash jarayonida (4) tengsizlikdan foydalaniladi. Kosinuslar teoremasiga ko‘ra,

$$a^2=b^2+c^2-2bc\cos A=(b-c)^2+2bc(1-\cos A)=(b-c)^2+4S\cdot\frac{1-\cos A}{\sin A}=(b-c)^2+4Stg\frac{A}{2}$$

U holda, $a^2+b^2+c^2=(b-c)^2+(c-a)^2+(a-b)^2+4S\left(tg\frac{A}{2}+tg\frac{B}{2}+tg\frac{C}{2}\right)\geq 4S\sqrt{3}$

tenglik $a=b=c$ bo‘lganda erishiladi.

9-usul. ABC uchburchakning AB tomonini absissa o‘qi bo‘ylab joylashtiramiz. AB kesmaning o‘rtasini esa koordinata boshi uchun qabul qilamiz. Natijada $A\left(-\frac{c}{2};0\right)$, $B\left(\frac{c}{2};0\right)$, $C(x;y)$ koordinatalarga ega bo‘lamiz. OY o‘qida $C_1\left(0;\frac{c\sqrt{3}}{2}\right)$ nuqtani olib, CC_1 masofani hisoblaymiz. U holda

$$CC_1^2=x^2+\left(\frac{c\sqrt{3}}{2}-y\right)^2=x^2+y^2+\frac{3c^2}{4}-c\sqrt{3}y=x^2+y^2+\frac{3c^2}{4}-2S\sqrt{3}$$

Ammo

$$a^2+b^2+c^2=BC^2+CA^2+AB^2=\left(x-\frac{c}{2}\right)^2+y^2+\left(x+\frac{c}{2}\right)^2+y^2+c^2=2\left(x^2+y^2+\frac{3c^2}{4}\right)=$$
$$=2CC_1^2+4S\sqrt{3}\geq 4S\sqrt{3}$$

Tenglik $C\equiv C_1$ bo‘lganda, ya’ni ABC muntazam uchburchak uchun o‘rinli bo‘ladi.

10-usul. Kompleks sonlarni tatbiq etamiz.

Qaralayotgan uchburchakning A,B,C uchlari mos ravishda $-1, 1$ va z affikslarga ega bo‘lsin. $A(-1),B(1),C(z)$ nuqtalarni qaraymiz. U holda

$AC=|z+1|$, $AC=|z-1|$, $AB=2$ bo‘lgani uchun

$$a^2+b^2+c^2=|z+1|^2+|z-1|^2+4=(z-1)(\overline{z}-1)+(z+1)(\overline{z}+1)+4=2z\overline{z}+6=2\left(|z|^2+3\right)$$

Ammo oriyentirlangan ABC uchburchak yuzi S:

$$S=\frac{i}{4}\begin{vmatrix}z_1 & \overline{z}_1 & 1\\ z_2 & \overline{z}_2 & 1\\ z_3 & \overline{z}_3 & 1\end{vmatrix}=\frac{i}{4}\begin{vmatrix}-1 & -1 & 1\\ 1 & 1 & 1\\ z & \overline{z} & 1\end{vmatrix}=\frac{i}{4}\cdot(-4iy)=y$$

Shuning uchun

$$a^2+b^2+c^2-4S\sqrt{3}=2\left(|z|^2+3\right)-4\sqrt{3}|y|\geq 2\left(|z|^2+3\right)-4\sqrt{3}|z|=2\left(|z|-\sqrt{3}^2\right)\geq 0$$

11-usul. Agar A,B,C uchburchak burchaklari bo‘lsa,

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\geq 2\sqrt{3}$$

tengsizlik o‘rinli. Bu tengsizlikning isboti $f(x)=\frac{1}{\sin x}\ (0<x<90^\circ)$ funksiyaning qabariqligidan kelib chiqadi. Haqiqatan ham, $f(x)$ qabariq funksiya bo‘lsa, $0<\alpha,\beta,\gamma<1$, $\alpha+\beta+\gamma=1$ shartni qanoatlantiruvchi sonlar uchun

$$f(\alpha x+\beta y+\gamma z)\leq\alpha f(x)+\beta f(y)+\gamma f(z)$$

tengsizlik bajariladi. Agar bu tengsizlikda $\alpha=\beta=\gamma=\frac{1}{3}$ deb olinsa, $f(x)=\frac{1}{\sin x}$ funksiya uchun

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\ge\frac{1}{\sin\dfrac{A+B+C}{3}}=3\frac{1}{\sin 60^\circ}=3\cdot\frac{2}{\sqrt{3}}=2\sqrt{3}$$

tengsizlikni yozamiz, Shunga ko‘ra

$$a^2+b^2+c^2\ge ab+bc+ca=2S\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)\ge 2\cdot 2\sqrt{3}=4\sqrt{3}$$

bo‘lib, tenglik $A=B=C$ bo‘lganda bajariladi.

12-usul. Bu usul ham qisman yuqoridagi usulni takrorlaydi:

$$\left(a^2-b^2\right)^2+\left(b^2-c^2\right)+\left(c^2-a^2\right)^2\ge 0$$

ma’lum tengsizlikdan

$$a^2+b^2+c^2\ge ab+bc+ca=2S\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)\ge 2\cdot 2\sqrt{3}=4\sqrt{3}$$

13-usul. Geometriyada (1) tengsizlikka nisbatan kuchliroq bo‘lgan Hadviger-Finsler tengsizligini

$$a^2+b^2+c^2\ge 4S\sqrt{3}+\left(a-b\right)^2+\left(b-c\right)^2+\left(c-a\right)^2$$

olishimiz mumkin.

Chap tomoni ko‘paytuvchilarga ajratilsa, tengsizlik

$$\left(a-b+c\right)\cdot\left(a+b-c\right)+\left(b-c+a\right)\cdot\left(b+c-a\right)+\left(c-a+b\right)\cdot\left(c+a-b\right)\ge 4S\sqrt{3}$$

ko‘rinishni oladi.

Endi

$$x=-a+b+c,\ y=a-b+c,\ z=a+b-c$$

belgilashlar kiritilsa, Geron formulasiga asosan,

$$4S\sqrt{3}=\sqrt{3(x+y+z)xyz}$$

bo‘lgani uchun, oxirgi tengsizlik

$$xy+yz+zx\ge\sqrt{3\left(x+y+z\right)xyz}$$

ko‘rinishni oladi. Bu tengsizlikning har ikkala qismini xyz ga bo‘lish natijasida ushbu tengsizlik

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ge\sqrt{3\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)}$$

ko‘rinishni oladi. So‘nggi tengsizlikda

$$u=\frac{1}{x},v=\frac{1}{y},w=\frac{1}{z}$$

almashtirishlar kiritilsa,

$$u+v+w\ge\sqrt{3\left(uv+vw+wu\right)}$$

tengsizlik hosil bo‘ladi. Bu tengsizlikni kvadratga ko‘tarib, oldindan to‘g‘riligi ma’lum bo‘lgan

$$u^2+v^2+w^2 \geq uv+vw+wu$$

tengsizlikka келamiz.

14-usul. Uchburchakning a,b,c tomonlari va S yuzi bo‘yicha

$$A=\frac{a^2+b^2+c^2}{S}\to \min$$

minimallashtirish masalasini qaraymiz. Sinuslar teoremasiga ko‘ra

$$a=2R\sin\alpha,\ b=2R\sin\beta,\ c=2R\sin\gamma$$

hamda uchburchakning yuzi

$$S=\frac{1}{2}bc\sin\alpha=2R^2\sin\alpha\sin\beta\sin\gamma$$

bo‘lgani uchun minimallashtirilayotgan ifoda

$$A=\frac{2\left(\sin^2\alpha+\sin^2\beta+\sin^2\gamma\right)}{\sin\alpha\sin\beta\sin\gamma}$$

ko‘rinishni oladi. O‘rta arifmetik va o‘rta geometrik miqdorlar orasidagi munosabatga ko‘ra bu ifodaning surati o‘zining eng kichik qiymatiga $\alpha=\beta=\gamma=\frac{\pi}{3}$ bo‘lganda erishadi, bu holatda maxraj o‘zining eng katta qiymatiga ega bo‘ladi. $\min A=4\sqrt{3}$ ekanidan (1) tengsizlikning o‘rinli ekanligi ravshan.

15-usul. Isbotlanishi lozim bo‘lgan tengsizlikda uchburchakning c tomonini kosinuslar teoremasiga ko‘ra $c^2=a^2+b^2-2ab\cos\gamma$ tenglik bilan, S yuzasini esa $S=\frac{1}{2}ab\sin\gamma$ tenglik bilan almashtirish natijasida bu tengsizlik

$$2\left(a^2+b^2\right)-2ab\cos\gamma\geq 2\sqrt{3}ab\sin\gamma$$

ko‘rinishni oladi. Endi bu tengsizlikning har ikkala qismini $2ab$ ga bo‘lib,

$$\frac{a}{b}+\frac{b}{a}\geq\sqrt{3}\sin\gamma+\cos\gamma$$

tengsizlikni hosil qilmiz. Oxirgi tengsizlik chap qismining eng kichik qiymati 2 ga, o‘ng qismining eng katta qiymati 2 ga teng bo‘lib, tenglik $a=b,\gamma=60^\circ$, ya’ni muntazam uchburchak uchun o‘rinli bo‘ladi. Oxirgi tengsizlikdan teskarisiga qaytish bilan isbotlanishi lozim bo‘lgan tengsizlikka kelishimiz mumkin.

Bu tengsizlikni isbotlashning bizga ma’lum bo‘lmagan boshqa usullari ham bo‘lishi mumkin.

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