THE CONSTRUCTING BOUNDARY VALUE PROBLEMS FOR DEFINED DISCONJUGATE'S INTERVAL LENGTH WITH MIXED CONSTRAYTENS

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Мақолада коэффициентлари турли чекланишларга эга бўлган иккинчи тартибли тенглама ечимининг қўшни ноллари орасидаги масофани аниқлаш учун чегаравий масала қурилган.

Калит сўзлар: *оптимал бошқарув назарияси, Понтрягин максимум принципи, чегаравий масала, тезкорлик вақти, аралаш чекланишга эга бўлган масала, ечим тебранмаслик масофаси*

В статье построена краевая задача для определения расстояния между соседними нулями нетривиального решения со смешанными ограничениями коэффициентов для уравнения второго порядка.

Ключевые слова: *теория оптимального управления, принцип максимум Понтрягина, краевая задача, время быстродействия, задача со смешанными ограничениями, расстояние неколебаемости решений.*

This article discusses the constructed boundary value problem to define distance between zeros of solution of the equation second order with coefficients mixed general restraints.

Key words: *theory optimum management, principle maximum Pontryagina, marginal problem, time of the speed, problem with mixed restrictions, distance unshakable decision.*

Introduction

A linear second order homogeneous differential equation

$$y'' + p_1(t)y' + p_2(t)y = 0 \qquad (1.1)$$

with real continuous coefficients $p_1(t)$, $p_2(t)$ is said to be disconjugate (Chebyshev nonoscillating) on real interval $[a,b] \subseteq R^1 = (-\infty, \div\infty)$, if no non-trivial solution $y(t)$ of this equation has at least one isolated root on this interval. A theorem C. de la Valee Poussin [1] asserts that any equation (1.1) is disconjugate on any interval sufficiently short with respect to the magnitude of the coefficients of the equation. More precisely, if

$$|p_1(t)| \le M_1, \quad |p_2(t)| \le M_2, \quad a \le t \le b$$

and the inequality

$$b - a < 2\int_0^{+\infty} \frac{ds}{s^2 + M_1 s + M_2}$$

holds, then equation (1.1) is disconjugate on $[a,b]$.

By using the Pontryagin Maximum Principle of the Control Theory we construct the boundary value problem for defined disconjugate's interval length for the equation (1.1) with more general restraints on the coefficients of the type

$$|p_1(t)| \le A, \quad \int_{-\infty}^{+\infty} p_2^2(t)\,dt \le \rho^2,$$

where $A \ge 0$, $\rho > 0$ are a given constants.

Main rezults

We consider control problems described by a linear homogeneous differential equation second order

$$y'' + p_1(t)y' + p_2(t)y = 0, \qquad (2.1)$$

where controllers $u_1(t), u_2(t)$ are measurable functions on R^1.

Let $\tilde{U}$ be the set of all vector functions $u(\cdot)=(u_1(\cdot),u_2(\cdot))$ such as the component functions $u_1(t), u_2(t)$ are Lebesque measurable on R^1 and satisfy the inequalities

$$| u_1(t) | \le A, \quad \int_{-\infty}^{+\infty} u_2^2(t)\,dt \le \rho^2, \tag{2.2}$$

where $A \ge 0, \quad \rho > 0$ - are given constants.

Function $y(t)$ is said to be a solution of (2.1), for a control vector $u=(u_1,u_2)\in U$, if $y(t)$ is of class continuous differential function, $y'(t)$ is absolutely continuous on R^1, and $y(t)$ satisfies equation (2.1) for almost all $t \in R^1$.

We will fix the initial point $t=0$ and denote by $T(u(\cdot))$ disconjugate's interval length for the control vector $u(\cdot)\in U$. Then it is clear that the disconjugate's interval length T for the equation (2.1) is

$$T = \inf\{T(u(\cdot)),\ u(\cdot)\in\tilde{U}\}. \tag{2.3}$$

It is noteworthy that $b-a$ of the interval $[a,b)\ (a<b)$ is a disconjugat's interval length for a class of the equation (1.1), where $u(\cdot)\in\tilde{U}$. Really, takes place the following:

Theorem1.(Analogue of an inequality Valle-Poussin). Let the differential equation (1.1) with measurable coefficients satisfying conditions (2.2) has nontrivial solution $y(t)$ having two zeros at the points $t=a$ and $t=b$, where $a<b$. Then the inequality is true

$$\frac{\rho(b-a)^{3/2}}{2h} + \frac{A(b-a)}{\pi} - 1 \ge 0 \tag{2.4}$$

Proof. Let $y(a)=y(b)=0$. Then from these equalities

$$y(t)=\int_a^t y'(s)ds,\ y(t)=-\int_t^b y'(s)ds$$

we have that

$$y(t)=\int_a^t y'(s)ds, \quad y(t)=-\int_t^b y'(s)ds,$$

$$|y(t)| \le \int_a^t |y'(s)|ds, \le \sqrt{t-a}\left(\int_a^t y'^{2}(s)ds\right)^{1/2}$$

$$|y(t)| \le \int_t^b |y'(s)|ds, \le \sqrt{b-t}\left(\int_t^b y'^{2}(s)ds\right)^{1/2}$$

Addition both inequalities and take an inequality

$$\sqrt{a_1b_1}+\sqrt{a_2b_2} \le \sqrt{(a_1+a_2)(b_1+b_2)},\ a_i \ge 0,\ b_i \ge 0,\ i=1,2,$$

we have

$$|y(t)| \le \frac{\sqrt{b-a}}{2}\left(\int_a^b y'^2(s)ds\right)^{1/2},\ a \le t \le b \tag{2.5}$$

Now multiplying both parts in the equation (2.1) on $y(t)$ and integrating on $[a,b]$, we get

$$\int_a^b y''(t)y(t)dt + \int_a^b u_1(t)y'(t)y(t)dt + \int_a^b u_2(t)y^2(t)dt = 0$$

Taking into account boundary conditions at the points $t = a$ and $t = b$: $y(a) = y(b) = 0$ we obtain

$$\int_a^b y'^2(t)dt = \int_a^b u_1(t)y'(t)y(t)dt + \int_a^b u_2(t)y^2(t)dt, \qquad (2.6)$$

Now by using Virtiger's inequality [4]:

$$\int_a^b y^2(s)ds \le \left(\frac{b-a}{\pi}\right)^2 \cdot \int_a^b y'^2(s)ds. \qquad (2.7)$$

and inequality (2.5) we estimated each integral of the right part of equality (2.6). Then

$$\left|\int_a^b u_1(t)y'(t)y(t)dt\right| \le A\int_a^b |y(t)|\cdot|y'(t)|dt \le A\left(\int_a^b y^2(t)dt\right)^{1/2} \cdot \left(\int_a^b y'^2(t)dt\right)^{1/2},$$

so that

$$\left|\int_a^b u_1(t)y'(t)y(t)dt\right| \le \frac{A(b-a)}{\pi}\int_a^b y'^2(t)dt. \qquad (2.8)$$

The second integral in (2.6) can be estimated in the following way:

$$\left|\int_a^b u_2(t)y^2(t)dt\right| \le \int_a^b |u_2(t)y(t)|\cdot|y(t)|dt \le \left(\int_a^b u_2^2(t)y^2(t)dt\right)^{1/2} \cdot \left(\int_a^b y^2(t)dt\right)^{1/2}.$$

Taking into account inequalities (2.5) and (2.7) from which we have

$$\left|\int_a^b u_2(t)y^2(t)dt\right| \le \frac{\rho(b-a)^{3/2}}{2\pi}\int_a^b y'^2(t)dt \qquad (2.9)$$

Now using an inequalities (2.8) and (2.9) from inequality (2.6) we obtain

$$\int_a^b y'^2(t)dt \le \left[\frac{\rho(b-a)^{3/2}}{2\pi} + \frac{A(b-a)}{\pi}\right]\cdot\int_a^b y'^2(t)dt \qquad (2.10)$$

Valid non triviality solution $y(t)$ equation (2.1),is clear that

$$\int_a^b y'^2(t)dt > 0.$$

Thus from (2.10) we obtain (2.4). Theorem is proved.

From the proved theorem follows, that the magnitude T for the class of the equations (2.1), where $u(\cdot) \in \tilde{U}$ is positive.

According to the proves theorem 1, the problem of search of length T of the maximal disconjugat's interval length for class of the equations (2.1) at $u(\cdot)\in\widetilde{U}$ is reduced to the following tame optimal control problem: for controlled object

$$\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = -u_2x_1 - u_1x_2,\end{cases}\quad u(\cdot)\in\widetilde{U} \tag{2.11}$$

with an initial condition $x(0)=(0,1)$ and terminal set $M=\{x\in R^2 : x_1=0, x_2<0\}$ it is necessary to find time of optimal time $\tau>0$ on the class of allowable control functions from $\widetilde{U}$.

There appears a question: whether there is even one admissible control function $u(\cdot)\in\widetilde{U}$, carrying out translation of a point $x(t)$ from an initial point (0,1) on the set M according to system (2.11)? The answer is positive. Really, we shall choose control function $u(t)$ as follows:

$$u_1(t)=0,\qquad t\in R^1$$

$$u_2(t)=\begin{cases}\left(\rho^4/\pi^2\right)^{1/3}, \ if\ \ t\in\left[0,\left(\pi^4/\rho^2\right)^{1/3}\right]\\ 0\quad , \ if\ \ t\notin\left[0,\left(\pi^4/\rho^2\right)^{1/3}\right]\end{cases}$$

Then the admissible solution of system (2.11) with the initial conditions $x_1(0)=0$, $x_2(0)=1$ looks like:

$$x_1(t)=\left(\frac{\pi}{\rho^2}\right)^{1/3}\sin\left[\left(\frac{\rho^2}{\pi}\right)^{1/3}\cdot t\right],\qquad x_2(t)=\cos\left[\left(\frac{\rho^2}{\pi}\right)^{1/3}\cdot t\right].$$

From at $t=\left(\frac{\pi^4}{\rho^2}\right)^{1/3}$ we have, that $x\left(\left(\frac{\pi^4}{\rho^2}\right)^{1/3}\right)=(0,-1)\in M$ and $\int\limits_{-\infty}^{+\infty}u_2{}^2(t)dt=\rho^2$, so that $u(\cdot)\in\widetilde{U}$**.** Thus, there is a control function $u(\cdot)\in\widetilde{U}$, for which $T(u(\cdot))<+\infty$.

Лемма 1. let $u(\cdot)\in\widetilde{U}$. Then for the solution $x(t)$ of system (2.11) with the initial conditions $x(0)=(0,1)$ at $0\le t\le h$ are following estimates:

$$\begin{aligned}&|x_1(t)|\le h\exp\left[h\left(A+\rho\sqrt{h}\right)\right],\\ &|x_2(t)|\le \exp\left[h\left(A+\rho\sqrt{h}\right)\right],\end{aligned} \tag{2.12}$$

is true.

Proof. First note that from (2.11) we have

$$\begin{aligned}&x_1(t)=\int_0^t x_2(s)ds,\\ &x_2(t)=1-\int_0^t\left[u_2(s)x_1(s)+u_1(s)x_2(s)\right]ds.\end{aligned} \tag{2.13}$$

It can be shown in the usual way that

$$\int_0^t u_2(s)x_1(s)ds=\int_0^t\left(\int_s^t u_2(\tau)d\tau\right)x_2(s)ds.$$

Therefore

$$x_2(t) = 1 - \int_0^t \left[u_1(s) + \int_s^t u_2(\tau) d\tau \right] x_2(s) ds$$

It now follows that

$$|x_2(t)| \le 1 + \int_0^t \left(A + \rho\sqrt{h-s} \right) |x_2(s)| ds, \quad 0 \le t \le h,$$

or

$$|x_2(t)| \le 1 + \left(A + \rho\sqrt{h} \right) \int_0^t |x_2(s)| ds, \quad 0 \le t \le h,$$

Bu using Gronuoll's inequality at $0 \le t \le h$ we have

$$|x_2(t)| \le \exp\left[h(A + \rho\sqrt{h}) \right] \qquad (2.14)$$

From (2.13) and (2.14) we get estimates (2.12) and lemma is proved.

Now by using lemma 1 we may proof the following theorem.

Theorema 2. In consideration of optimal control problem there exists measurable time optimal controller $u(\cdot) \in \tilde{U}$. which translates the trajectory appropriate to the system (2.11) of a point (0,1) on terminal set M in the minimal time $\tau > 0$.

Proof. Select a minimization sequence $(x^k(\cdot),\ u^k(\cdot))$, $k = 1,2,3.....$ where $u(\cdot) \in \tilde{U}$; $x_1^{\,k}(0) = 0,\ x_2^{\,k}(0) = 1;\ x_1^{\,k}(t^k) = 0$, $x_2^{\,k}(t^k) < 0$, $t^k -$ a times sequence decreasing monotonically at $k \to +\infty$ converges to $\tau > 0 -$ infimum of the investigates functional at above considered statement. By the lemma 1, at $0 \le t \le \tau$ sequence $\{x^k(t)\}$, $k = 1,2,...$ is uniformly bounded and for any $t_1, t_2, \in [0, \tau]$, where $t_1 \le t_2$ is holds

$$\left| x_2^{\,k}(t_2) - x_2^{\,k}(t_1) \right| = \left| \int_{t_1}^{t_2} \dot{x}_2^{\,k}(s) ds \right| \le \int_{t_1}^{t_2} \left| u_1^{\,k}(s) x_2^{\,k}(s) + u_2^{\,k}(s) x_1^{\,k}(s) \right| ds \le$$

$$\le A \int_{t_1}^{t_2} \left| x_2^{\,k}(s) \right| ds + \rho \left(\int_{t_1}^{t_2} \left| x_1^{\,k}(s) \right|^2 ds \right)^{1\backslash 2} \le A M_0 (t_2 - t_1) + \rho M_0 \sqrt{t_2 - t_1},$$

and

$$\left| x_1^{\,k}(t_2) - x_1^{\,k}(t_1) \right| = \left| \int_{t_1}^{t_2} \dot{x}_1^{\,k}(s) ds \right| \le \int_{t_1}^{t_2} \left| x_2^{\,k}(s) \right| ds \le M_0 (t_2 - t_1),$$

where $k = 1,2,....,\ M_0 = \exp\left[\tau \left(A + \rho\sqrt{\tau} \right) \right]$. That the sequence $\{x^k(t)\}$ is equally continuous. According to the Ascoli's theorem , from that it is possible to select a subsequence solution's , which uniformly converging to some continuous function $x^*(t)$ on the segment $[0, \tau]$ of solutions of the system (2.11). Without restriction of a generality we can consider that sequence $\{x^k(t)\}$ uniformly converges on $[0, \tau]$ to $x^*(t)$, besides

$$x_1^*(0) = 0, x_2^*(0) = 1; x_1^*(\tau) = 0, x_2^*(\tau\ ) \le 0 \cdot$$

Now we show, that $x^*(t)$ is a solution of the system (2.12) at some admissible controller $u^*(t)$ on $[0, \tau]$.

At $t\ \in [0, \tau]$ we have:

$$x_2^*(t) = \lim_{k \to +\infty} x_2^k(t) = 1 - \lim_{k \to +\infty} \int_0^t \left[u_1^k(s) x_2^k(s) + u_2^k(s) x_1^k(s) \right] ds =$$

$$= 1 - \lim_{k \to \infty} \left[\int_0^t \left(x_2^k(s) - x_2^*(s) \right) u_1^k(s) ds + \int_0^t \left(x_1^k(s) - x_1^*(s) \right) u_2^k(s) ds + \int_0^t \left(u_1^k(s) x_2^*(s) + u_2^k(s) x_1^*(s) \right) ds \right].$$

As

$$\left|\int_0^t \left(x_2^k(s)-x_2^*(s)\right)u_1^k(s)ds\right| \le A\tau \max_{t\in[0,\tau]}\left|x_2^k(t)-x_2^*(t)\right|,$$

$$\left|\int_0^t \left(x_1^k(s)-x_1^*(s)\right)u_2^k(s)ds\right| \le \rho\sqrt{\tau} \max_{t\in[0,\tau]}\left|x_1^k(t)-x_1^*(t)\right|,$$

then

$$x_2^*(t)=1-\lim_{k\to+\infty}\int_0^t\left[u_1^k(s)x_2^*(s)+u_2^k(s)x_1^*(s)\right]ds$$

As the set of all functions $\left\{u^k(\cdot)\right\}$ belong to some sphere of space L_2, тhen from that it is possible to select weak convergence subsequence. We shall count , that sequence $\left\{u^k(\cdot)\right\}$ is weak convergence to some function $u^*(\cdot)$. As has been mentioned, we have

$$\lim_{k\to+\infty}\int_0^t\left[u_1^k(s)-u_1^*(s)\right]x_2^*(s)ds=0,$$

$$\lim_{k\to+\infty}\int_0^t\left[u_2^k(s)-u_2^*(s)\right]x_1^*(s)ds=0.$$

Now it is not difficult to see that functions $x_1^*(t)$, $x_2^*(t)$ are satisfied equation of the system:

$$x_1^*(t)=\int_0^t x_2^*(s)ds,$$

$$x_2^*(t)=1-\int_0^t\left[u_2^*(s)x_1^*(s)+u_1^*(s)x_2^*(s)\right]ds,$$

which is equivalent (2.11).

Similarly [1,2] may can show , that $\left|u_1^*(t)\right|\le A$ almost everywhere on $[0,\tau]$ and $\int_0^\tau u_2^{*2}(t)dt\le\rho^2$. As the equation (2.11) is linear homogenous vector equation at $u_1=u_1^*(t), u_2=u_2^*(t)$, тhen $x_2^*(\tau)<0$ and hence, $x^*(\tau)\in M$. Theorem is proved.

It is not difficult to prove the following theorem:

Theorem 3. Let τ-time for which optimal controller translates the trajectory appropriate to the system (2.11) of a point (0,1) on terminal set M. Then the equality takes place: $\grave{O}=\tau$.

Using the weak variation it is possible to show that the principle of a maximum for the formulated above task optimum speed is valid. Now by using a Pontryagin's maximum principle [1], we deduce analogue of the Eyler equations from variational calculation for the considered problem of the disconjugates theory. For definition of T for a class of the equations (1.1) at $u(\cdot)\in\widetilde{U}$ two points value problem are constructed.

Theorem 4. Length T of the maximal disconjugat's interval length for the class equations (1.1) at $u(\cdot)\in\grave{U}$ is characterized by solving the following boundary value problems:

$$y''+Ay'+\frac{\rho\exp(At)y^3}{\left(2\int_0^{T/2}\exp(2As)y^4(s)ds\right)^{1/2}}=0, \qquad (2.16)$$

$$y(0)=0, y'(0)=0, y'(T/2)=0. \qquad (2.17)$$

Proof. Let's put a new variable

$$x_3(t) = \int_0^t u_2^2(\tau)d\tau .$$

Then it is clear, that

$$\dot{x}_3 = u_2^2, x_3(0) = 0, x_3(T) \le \rho^2,$$

and the formulated above time optimal control problem is equivalent to the following problem: to find admissible controller $\tilde{u}(\cdot) \in \tilde{U}$ such that the decision appropriate to him solution $\tilde{x}(t)$ of the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -u_2 x_1 - u_1 x_2 \\ \dot{x}_3 = u_2^2 \end{cases} \qquad (2.18)$$

with the entry conditions $\tilde{x}(0) = (0,1,0)$ gets on the set

$$\tilde{M} = \left\{ x \in R^3 : x_1 = 0, x_2 < 0, x_3 \le \rho^2 \right\}$$

For the least time $\tau > 0$,which coincides with T. By the Pontryagin's maximum principle for an optimality controller $u^*(t), 0 \le t \le T$, it is necessary the existence such absolutely continuous vector functions $\tilde{\psi}(t) = (\psi_1(t), \psi_2(t), \psi_3(t))$ and constant $\psi_0 \le 0$, that first vector $(\psi_0, \tilde{\psi}(t))$ is not zero for all $t \in [0,T]$, where $\psi_1(t), \psi_2(t), \psi_3(t)$-are solutions of adjoint system

$$\begin{cases} \dot{\psi}_1 = u_2^*(t)\psi_1 \\ \dot{\psi}_2 = u_1^*(t)\psi_2 - \psi_1 \\ \dot{\psi}_3 = 0 \end{cases} \qquad (2.19)$$

and secondly almost every where on $[0,T]$ hold maximum condition

$$\sup_{\substack{|u_1| \le A \\ u_2 \in R^1}} H(\psi_0, \tilde{\psi}(t), \tilde{x}^*(t), u) = H(\psi_0, \tilde{\psi}(t), \tilde{x}^*(t), u^*(t)) \equiv 0 , \qquad (2.20)$$

where

$$H(\psi_0, \tilde{\psi}, u, \tilde{x}) = \psi_0 + \psi_1 x_2 - \psi_2(u_2 x_1 + u_1 x_2) + \psi_3 u_2^2$$

Besides hold a transversality's conditions on the right end of a trajectory $\tilde{x}^*(t)$:

a) $\psi_2(T) = \psi_3(T) = 0$, if $x_3^*(T) < \rho^2$,

b) $\psi_2(T) = 0$, if $x_3^*(T) = \rho^2$.

Using the equation (2.18), (2.19), almost every where on [0,T] we have:

$$\frac{d}{dt} < \tilde{x}^*(t), \tilde{\psi}(t) >= \frac{d}{dt}\left(x_1^*(t)\psi_1(t) + x_2^*(t)\psi_2(t) + x_3^*(t)\psi_3(t) \right) = \psi_3(t)u_2^{*2}(t)$$

Further, at $\dot{\psi}_3 = 0$, we have $\psi_3(t) \equiv \mu = const$.
Consequently almost every where on $[0,T]$

$$\frac{d}{dt}\left(x_1^*(t)\psi_1(t) + x_2^*(t)\psi_2(t) + \mu x_3^*(t) \right) = \mu u_2^{*2}(t)$$

After integration we have

$$x_1^*(t)\psi_1(t) + x_2^*(t)\psi_2(t) \equiv C, \quad 0 \le t \le T.$$

Having put $t = T$, with the account $x_1^*(T) = \psi_2(T) = 0$,we receive, that

$$x_1^*(t)\psi_1(t) + x_2^*(t)\psi_2(t) \equiv 0, \quad 0 \le t \le T \quad (2.21)$$

The maximum in (2.20) on variable u_2 reached, if holds where the necessary condition of extremum $\partial H / \partial u_2 = 0$, that is

$$x_1^*(t)\psi_1(t) + x_2^*(t)\psi_2(t) \equiv 0, \quad 0 \le t \le T. \quad (2.22)$$

Let's consider a case: a) $x_3^*(T) < \rho^2$. In this case it is not difficult to see that $\psi_3(t) \equiv 0$, and from (2.22) follows that $\psi_2(t) \equiv 0$, and it entails by (2.21) $\psi_1(t) \equiv 0$. Then from (2.20) follows, that $\psi_0 = 0$. Thus, $\psi_0 = \psi_1(t) = \psi_2(t) = \psi_3(t) \equiv 0$, that contradiction non triviality of a vector $(\psi_0, \tilde{\psi}(t))$.

Thus it is shown, that

$$\int_0^\tau u_2^{*2}(t)dt = \rho^2 \quad (2.23)$$

The above-stated reason show that $\psi_3(t) \equiv \mu \neq 0$.

The maximum in (2.20) on variable u_2 is reached, when $\mu < 0$. From (2.22) we find to a component $u_2^*(t)$ of optimal controller $u^*(t)$, which looks like

$$u_2^*(t) = \frac{\psi_2(t)x_1^*(t)}{2\mu} \quad (2.24)$$

(here in after optimal controller $u_2^*(t)$ is subjected to reorganization on the set of a measure zero so, that $u_2^*(t)$ became continuous function).

Let's prove that $\psi_0 \neq 0$. Let assume opposite: $\psi_0 = 0$.Then from (2.20) and (2.24) at $t = T$ follows that $\psi_1(T)x_2^*(T) = 0$. But as $x_2^*(T) < 0$, then $\psi_1(T) = 0$. Now from $\psi_1(T) = \psi_2(T) = 0$ and (19) we shell perform , then $\psi_1(T) = \psi_2(T) = 0$, $0 \le t \le T$. Then from (24) follows , that $u_2^*(t) \equiv 0$, $0 \le t \le T$. Having received the contradiction with that $\int_0^\tau u_2^{*2}(t)dt = \rho^2 > 0$. Hence $\psi_0 \neq 0$ and we can put $\psi_0 = -1$. Further at $t = 0$ from (2.21) follows that $\psi_2(0) = 0$,and the turn from (2.20) follows $\psi_1(0) = 1$.

Now we shall prove, that $\psi_2(t) \neq 0$, $t \in (0,T)$. Suppose opposite: let $\psi_2(\xi) = 0$ for some $\xi \in (0,T)$.Then from (2.21) at $t = \xi$ we have $\psi_1(\xi) = 0$. Consequently, $\psi_1(T) = \psi_2(T) \equiv 0$, $t \in [0,T]$. From and (2.24) we have $u_2^*(t) \equiv 0$, $t \in [0,T]$. Having received the contradiction to that $\int_0^\tau u_2^2(t)dt = \rho^2 > 0$.Thus the function $\psi_2(t)$ saves its sign on the interval (0,T).

From $\psi_2(0) = 0$, $\psi_1(0) = 1$ and from the second equation of the system (2.19) follows , that

$$\psi_2(t) < 0, \; 0 < t < T. \quad (2.25)$$

Correlations (2.24) and (2.25) show that $u_2^*(t) > 0$,. $t \in (0,T)$ and $u_2^*(t)$ is continuous functions on $[0,T]$ (after reorganization on the set of a measure zero). Then aggress lemma 1, derivative $y'(t) = x_2^*(t)$ has only one zero on the interval (0,T). Designate it through by θ, so that $y'(\theta) = x_2^*(\theta) = 0$.

From the above-stated reasons it is clear , that maximum on variable u_1 in the ratio (2.20) is achieved on piecewise smooth function:

$$u_1^*(t) = A\operatorname{sgn} x_2^*(t), \; 0 \le t \le T \quad (2.26)$$

(there and after $u_1^*(t)$ reorganization on the set of a measure zero). From (2.24), (2.26), (2.12), (2.19) we have

$$\begin{cases} \dot{x}_1^* = x_2^*, \\ \dot{x}_2^* = -u_2^*(t)x_1^* - Ax_2^*, \quad 0 \le t < \theta \end{cases} \tag{2.27}_1$$

$$\begin{cases} \dot{x}_1^* = x_2^*, \\ \dot{x}_2^* = -u_2^*(t)x_1^* + Ax_2^*, \quad \theta < t \le T \end{cases} \tag{2.27}_2$$

$$\begin{cases} \dot{\psi}_1 = u_2^*(t)\psi_2, \\ \dot{\psi}_2 = A\psi_2 - \psi_1, \quad 0 \le t < \theta \end{cases} \tag{2.28}_1$$

$$\begin{cases} \dot{\psi}_1 = u_2^*(t)\psi_2, \\ \dot{\psi}_2 = -A\psi_2 - \psi_1, \quad \theta < t \le T \end{cases}. \tag{2.28}_2$$

From the (2.27) and (2.28) by continuous function $u_2^*(t)$, $0 \le t \le T$ it is clear , that functions $\psi_2(t)$ and $x_2^*(t)$ piecewise smooth , and they have continuous derivative on $[0,T]$, the except at the point $t = \theta$. Therefore from (2.27),(2.28) follows that

$$\ddot{x}_1^* + A\dot{x}_1^* + u_2^*(t)x_1^* = 0, \quad 0 \le t < \theta, \tag{2.29}_1$$

$$\ddot{\psi}_2 - A\dot{\psi}_2 + u_2^*(t)\psi_2 = 0, \quad 0 < t < \theta \tag{2.29}_2$$

and

$$\ddot{x}_1^* - A\dot{x}_1^* + u_2^*(t)x_1^* = 0, \quad \theta < t \le T, \tag{2.30}_1$$

$$\ddot{\psi}_2 + A\dot{\psi}_2 + u_2^*(t)\psi_2 = 0, \quad \theta < t \le T. \tag{2.30}_2$$

To multiplying $(2.29)_1$ on $\psi_2(t)$, and $(2.29)_2$ on $x_1^*(t)$ and summing this equalities we have:

$$\ddot{x}_1^*\psi_2 - x_1^* \cdot \ddot{\psi}_2 + A(\dot{x}_1^*\psi_2 + x_1^*\dot{\psi}_2) = 0, \quad 0 \le t < \theta. \tag{2.31}_1$$

Similar from $(2.30)_1$ and $(2.30)_2$ we get

$$\ddot{x}_1^*\psi_2 - x_1^* \cdot \ddot{\psi}_2 - A(\dot{x}_1^*\psi_2 + x_1^*\dot{\psi}_2) = 0, \quad \theta < t \le T. \tag{2.31}_2$$

After integration the equations $(2.31)_1$ and $(2.31)_2$ taking into consideration boundary values of functions $x_1^*(t)$, $\psi_2(t)$, we have соответственно:

$$\dot{x}_1^*\psi_2 - x_1^* \cdot \dot{\psi}_2 + Ax_1^*\psi_2 = 0, \quad 0 \le t < \theta \tag{2.32}_1$$

and

$$\dot{x}_1^*\psi_2 - x_1^* \cdot \dot{\psi}_2 - Ax_1^*\psi_2 = 0, \quad \theta < t \le T \tag{2.32}_2$$

From $(2.32)_1$ and $(2.32)_2$ after integration we get, that

$$\psi_2(t) = !\ {}_1x_1^*(t)e^{At}, \quad 0 \le t < \theta, \qquad (2.33)_1$$

and

$$\psi_2(t) = !\ {}_2x_1^*(t)e^{-At}, \quad \theta < t \le T. \qquad (2.33)_2$$

As $\psi_2(t)$ is absolutely continuous function, that C_1 and C_2 are connected with equality $C_2 = C_1e^{2A\theta}$. Differentiating $(2.33)_1$, we have:

$$\dot{\psi}_2(t) = (!\ {}_1\dot{x}_1^*(t) + AC_1x_1^*(t))e^{At}$$

From that taking into consideration

$$\dot{\psi}_2(0) = -\psi_1(0) = -1, x_1^*(0) = 0, \dot{x}_1^*(0) = x_2^*(0) = 1, \text{ we get } C_1\text{=-1.}$$

Thus

$$\psi_2(t) = \begin{cases} -x_1^*(t)e^{At}, & 0 \le t \le \theta, \\ -x_1^*(t)e^{-A(t-2\theta)}, & \theta \le t \le T \end{cases}.$$

Having substituted expression $u_2^*(t)$ into (2.24), we receive

$$u_2^*(t) = -\frac{1}{2\mu}\begin{cases} x_1^{*2}(t)e^{At}, & 0 \le t \le \theta, \\ x_1^{*2}(t)e^{-A(t-2\theta)}, & \theta \le t \le T \end{cases} \qquad (2.34)$$

According to (2.26), (2.34) the equation (2.1) (after replacement variable $x_1^* = y$, $x_2^* = y'$) looks like

$$y'' + Ay' - \frac{1}{2\mu}e^{At}y^3 = 0, \quad 0 \le t \le \theta,$$

$$y'' - Ay' - \frac{1}{2\mu}e^{-A(t-2\theta)}y^3 = 0, \quad \theta \le t \le T. \qquad (2.35)$$

Let's put function $\varphi(t) = y(t+\theta)$. Then (2.35) will accept type:

$$\phi'' + A\phi' - \frac{1}{2\mu}e^{A(t+\theta)}\phi^3 = 0, \quad -\theta \le t \le 0,$$

$$\phi'' - A\phi' - \frac{1}{2\mu}e^{-A(t-\theta)}\phi^3 = 0, \quad 0 \le t \le T-\theta, \qquad (2.36)$$

and, moreover

$$\phi'(T-\theta) < 0, \phi(-\theta) = 0, \phi'(-\theta) = 1, \phi'(0) = 0, \phi(T-\theta) = 0\,. \qquad (2.37)$$

Having united both equation (36) in one , we have:

$$\phi'' + A|\phi'| - \frac{1}{2\mu}e^{A(\theta-|t|)}\phi^3 = 0, \quad -\theta \le t \le T-\theta\ . \qquad (2.38)$$

Now we perform, that $T = 2\theta$.

We see, that the solution $\varphi(t)$ of the equation (2.38) is symmetric concerning an axis $t = 0$. Certainly if $\varphi = \chi(t)$ the solution of the equation (2.38), then $\varphi = \chi(-t)$ also is the solution of the equation (2.38). Therefore a positive solution $\varphi(t)$ of boundary value problem (2.38), (2.37) at $t \ge 0$ satisfies also the following

boundary conditions: $\varphi(\theta)=0,\ \varphi'(\theta)=1$. From reasons of symmetry follows, that $T-\theta=\theta$, that is $T=2\theta$.

It is necessary to define unknown constant $\mu<0$. By force replacement $\varphi(t)=y(t+\theta)$ from (2.34) we have:

$$v(t)=u_2^*(t+\theta)=-\frac{e^{A\theta}}{2\mu}\begin{cases}\phi^2(t)e^{At}, & -\theta\le t\le 0,\\ \phi^2(t)e^{-At}, & 0\le t\le\theta.\end{cases}$$

Therefore aggress to (2.23) we shell be receive:

$$\rho^2=\int_0^T u_2^{*2}(t)dt=\int_{-\theta}^{\theta}u_2^*(t+\theta)^2dt=\int_{-\theta}^{\theta}v^2(t)dt=\frac{e^{2A\theta}}{4\mu^2}\int_{-\theta}^{\theta}e^{-2A|s|}ds.$$

Hence it follows

$$-\frac{1}{2\mu}=\frac{\rho\exp(-A\theta)}{\left(\int_{-\theta}^{\theta}\exp(-2A|s|)\phi^4(s)ds\right)^{1/2}}. \qquad (2.39)$$

From (2.38) and (2.39) follows, that function $\varphi(t)$ satisfies to the following equation:

$$\phi''+\ |\phi'|+\frac{\rho\exp(-A|t|)\phi^3}{\left(\int_{-\theta}^{\theta}\exp(-2A|s|)\phi^4(s)ds\right)^{1/2}}=0, \qquad (2.40)$$

with boundary conditions

$$\phi(-\theta)=0,\phi'(-\theta)=1,\phi'(0)=0,\phi(\theta)=0,\phi'(\theta)=-1 \qquad (2.41)$$

From the above mentioned analysis follows, that it is enough to consider a boundary value problem (2.40), (2.41) on the segment $-\theta\le t\le 0$. Hence it follows, as it is to see the statement of the theorem .

Theorem 5. Let Λ-set of numbers $\gamma>0$ for which can be solved boundary value problem (2.16),(2.17) (with replacement T to γ).Then hold an equality: $\min\{\gamma:\gamma\in\Lambda\}=T$.

Proof. For demonstration, that $\inf\{\gamma:\gamma\in\Lambda\}>0$, we consider the following boundary value problem (with replacement T to γ) on the segment $[-\gamma/2,\gamma/2]$:

$$\phi''+A|\phi'|+\frac{\rho\exp(-A|t|)\phi^3}{\left(2\int_{-\gamma/2}^{0}\exp(-2A|s|)\phi^4(s)ds\right)^{1/2}}=0, \qquad (2.42)$$

$$\phi(-\gamma/2)=\phi(\gamma/2)=0,\phi'(-\gamma/2)=-\phi'(\gamma/2)=1,\phi'(0)=0. \qquad (2.43)$$

At $\gamma\in\Lambda$ the boundary value problem (2.42), (2.43) have a solution by virtue of the theorem of existence and uniqueness and following property: if $y(t)$ is the solution of the boundary value problem (2.16), (2.17) at T=γ on the segment $[0,\gamma/2]$, then function $y(-t)$ on the segment $[-\gamma/2,0]$ satisfies to the equation

$$y''-Ay'+\frac{\rho\exp(-At)y^3}{\left(2\int_{-\gamma/2}^{0}\exp(-2A|s|)y^4(s)ds\right)^{1/2}}=0,$$

and boundary conditions $y'(\gamma/2)=0,y(\gamma)=0,y'(\gamma)=-1$.

Multiplying both parts of the equation (2.42) on $\varphi(t)$ and integrating on the segment $[-\gamma/2,\gamma/2]$, we have:

$$\int_{-\gamma/2}^{\gamma/2}\phi'^2(s)ds = A\int_{-\gamma/2}^{\gamma/2}\phi(s)\left|\phi'(s)\right|ds + \frac{\rho\cdot\int_{-\gamma/2}^{\gamma/2}\exp(-A|s|)\phi^4(s)ds}{\left(\int_{-\gamma/2}^{\gamma/2}\exp(-2A|s|)\phi^4(s)ds\right)^{1/2}}$$

Using inequalities (2.6), (2.7) from that the simple calculations we shell obtain

$$\frac{\rho\gamma^{3/2}}{2\pi}+\frac{A\gamma}{\pi}-1\geq 0.$$

From that $\gamma\geq\xi_0$,where ξ_0- a positive root of the equation

$$\frac{\rho\gamma^{3/2}}{2\pi}+\frac{A\gamma}{\pi}-1=0\cdot$$

Hence, $\inf\{\gamma:\gamma\in\Lambda\}\geq\xi_0>0$.

Let's assume, that $\inf\{\gamma:\gamma\in\Lambda\}=\gamma_0<T$ and there is $\gamma_1\in\Lambda$ such that $\gamma_1<T$. For γ_1 this responds solution $\tilde{y}(t)$ of the boundary value problem (2.16), (2.17) on the segment $[0,\gamma/2]$. Let's consider a pair function:

$$\tilde{u}_1(t)=\begin{cases}A, & if \quad 0\leq t\leq\gamma_{1/2},\\ -A, & if \quad \gamma_{1/2}\leq t\leq\gamma_1,\\ 0, & if \quad t\notin[0,\gamma_1]\end{cases}$$

$$\tilde{u}_2(t)=\begin{cases}\dfrac{\rho\exp(At)\tilde{y}^2(t)}{\left(2\int_0^{\gamma_{1/2}}\exp(2As)\tilde{y}^4(s)ds\right)^{1/2}}, & if \quad 0\leq t\leq\gamma_{1/2},\\ \dfrac{\rho\exp(-At)\tilde{y}^2(t)}{\left(2\int_{\gamma_{1/2}}^{\gamma_1}\exp(-2As)\tilde{y}^4(s)ds\right)^{1/2}}, & if \quad \gamma_{1/2}\leq t\leq\gamma_1,\\ 0, & if \quad t\notin[0,\gamma_1]\end{cases}$$

where the solution $\tilde{y}(t)$ is continued on a segment $[\gamma_{1/2},\gamma_1]$ by means of equality $\tilde{y}(t)=y(\gamma_1-t)$.

Now we shall prove, that functions $\tilde{u}_1(t)$, $\tilde{u}_2(t)$ at $t\in R^1$ are allowable functions in a considered problem of disconjugat's theory. We have:

$$\left|\tilde{u}_1(t)\right|\leq A,\ t\in R^1;$$

$$\int_{-\infty}^{\infty}\tilde{u}_2^2(t)dt=\int_0^{\gamma_1}\tilde{u}_2^2(t)dt=\int_0^{\gamma_{1/2}}\tilde{u}_2^2(t)dt+\int_{\gamma_{1/2}}^{\gamma_1}\tilde{u}_2^2(t)dt=\int_0^{\gamma_{1/2}}\frac{\rho^2\exp(2At)\tilde{y}^4(t)dt}{\left[2\int_0^{\gamma_{1/2}}\exp(2As)\tilde{y}^4(s)ds\right]}+$$

$$+\int_{\gamma_{1/2}}^{\gamma_1}\frac{\rho^2\exp(-2At)\tilde{y}^4(t)dt}{\left[2\int_{\gamma_{1/2}}^{\gamma_1}\exp(-2As)\tilde{y}^4(s)ds\right]}\le\frac{\rho^2}{2}+\frac{\rho^2}{2}=\rho^2.$$

Hence functions $\tilde{u}_1(t)$, $\tilde{u}_2(t)$ are allowable pair, that is $\tilde{u}(\cdot)=(\tilde{u}_1(\cdot)\ \tilde{u}_2(\cdot)\ \in\tilde{U}$. But then there should be $\gamma_1>T$. The received contradiction proves the theorem.

If in considered problem of the theory disconjugat's instead of a class of measurable functions $\tilde{U}$ to consider a class measurable functions $u(\cdot)\in\tilde{U}_\alpha\ (\alpha>1)$, where

$$\tilde{U}_\alpha:|u_1(t)|\le A,\quad\int_{-\infty}^{+\infty}|u_2(t)|^\alpha\,dt\le\rho^2,t\in R^1 \tag{2.43}$$

That by using similar relations stated above (with the help, of a stated above maximum), it is possible to deduce the following ratio:

$$\mu\alpha\left|u_2^*(t)\right|^{\alpha-1}\operatorname{sgn}u_2^*(t)=x_1^*(t)\psi_2(t),\quad 0\le t\le T. \tag{2.44}$$

Further, similarly disassembled means happen $\alpha=2$ is shown, that $T=2\theta$, where $\theta>0$ and $x_2^*(\theta)=0$. Thus (with to account $x_1=y,x_2=y'$):

$$u_1^*(t)=A\operatorname{sgn}y'(t),\quad 0\le t\le T,$$

$$u_2^*(t)=\frac{1}{|\mu\alpha|^{\frac{1}{\alpha-1}}}\begin{cases}\exp\left(\dfrac{At}{\alpha-1}\right)y^{\frac{2}{\alpha-1}}(t),\quad 0\le t\le T/2,\\ \exp(-\dfrac{A(t-T)}{\alpha-1})y^{\frac{2}{\alpha-1}}(t),\dfrac{T}{2}\le t\le T\end{cases}. \tag{2.45}$$

Further unknown constant $\mu<0$ is defined from ratio

$$\int_0^T\left|u_2^*(t)\right|^\alpha dt=\rho^2$$

We have:

$$\rho^\alpha=\frac{1}{|\mu\alpha|^{\alpha/\alpha-1}}\left[\int_0^{T/2}e^{\frac{\alpha At}{\alpha-1}}y^{\frac{2\alpha}{\alpha-1}}(t)dt+\int_{T/2}^{T}e^{-\frac{\alpha A(t-T)}{\alpha-1}}y^{\frac{2\alpha}{\alpha-1}}(t)dt\right]=$$

$$=\frac{1}{|\mu\alpha|^{\alpha/(\alpha-1)}}\left[\int_0^{T/2}e^{\frac{\alpha At}{\alpha-1}}y^{\frac{2\alpha}{\alpha-1}}(t)dt+\int_0^{T/2}e^{\frac{\alpha At}{\alpha-1}}y^{\frac{2\alpha}{\alpha-1}}(T-t)dt\right].$$

By virtue of symmetry of the solution $y(t)$, concerning a straight line $t=T/2$, finally we obtain

$$\left(\frac{1}{|\mu\alpha|}\right)^{\frac{1}{\alpha-1}}=\frac{\rho}{\left(2\int_0^{T/2}\exp\left(\dfrac{\alpha As}{\alpha-1}\right)y^{\frac{2\alpha}{\alpha-1}}(s)ds\right)^{1/\alpha}}=\frac{\rho}{\left(2\int_{T/2}^{T}\exp\left(-\dfrac{\alpha A(s-T)}{\alpha-1}\right)y^{\frac{2\alpha}{\alpha-1}}(s)ds\right)^{1/2}}.$$

$$u_2^*(t)=\begin{cases}\dfrac{\rho\exp\left(At/(\alpha-1)\right)y^{2/(\alpha-1)}(t)}{\left(2\int\limits_0^{T/2}\exp\left(\alpha As/(\alpha-1)\right)y^{2\alpha/(\alpha-1)}(s)ds\right)^{1/\alpha}}, & 0\le t\le T/2,\\ \dfrac{\rho\exp\left(At/(\alpha-1)\right)y^{2/(\alpha-1)}(t)}{\left(2\int\limits_{T/2}^{T}\exp\left(-\alpha As/(\alpha-1)\right)y^{2\alpha/(\alpha-1)}(s)ds\right)^{1/\beta}}, & \dfrac{T}{2}\le t\le T\end{cases} \tag{2.46}$$

On the segment [0,T/2] boundary value problem for $y(t)$ has view:

$$y''+Ay'+\frac{\rho\exp\left(At/(\alpha-1)\right)y^{(\alpha+1)/(\alpha-1)}}{\left(2\int\limits_0^{T/2}\exp(\alpha As/(\alpha-1))y^{2\alpha/(\alpha-1)}(s)ds\right)^{1/\alpha}}=0,$$

$$y(0)=0,\ y'(0)=1,\ y'(T/2)=0, \tag{2.47}$$

and on the segment [T/2,T] the following view:

$$y''-Ay'+\frac{\rho\exp\left(-At/(\alpha-1)\right)y^{(\alpha+1)/(\alpha-1)}}{\left(2\int\limits_{T/2}^{T}\exp(-\alpha As/(\alpha-1))y^{2\alpha/(\alpha-1)}(s)ds\right)^{1/\alpha}}=0,$$

$$y(T)=0,\ y'(T)=-1,\ y'(T/2)=0.$$

Thus the following theorem is true.

Theorem 6. Length T maximal disconjugate's interval length of the class equations (2.1) at $u(\cdot)\in \acute{U}_\alpha\ \alpha>1)$ characterized , that solved boundary value problem (2.47).

As an example we calculate size T in a case , when A=0. This case, the result coincides with the Y.Melentsova's result [5]:

$$T=\left[\frac{\alpha^\alpha}{(2\alpha-1)(\alpha-1)^{\alpha-1}\rho^\alpha}\right]^{\frac{1}{2\alpha-1}}\cdot\left(2\int\limits_0^1\frac{ds}{\sqrt{1-s^{\frac{2\alpha}{\alpha-1}}}}\right)^{\frac{2\alpha}{2\alpha-1}} \tag{2.49}$$

Indeed at A=0 boundary value problem (2.47) have view:

$$y''+\frac{\rho y^{\frac{\alpha+1}{\alpha-1}}}{\left(2\int\limits_0^{T/2}y^{\frac{2\alpha}{\alpha-1}}(s)ds\right)^{1/\alpha}}=0, \tag{2.50}$$

$$y(0)=0,\ y'(0)=1,\ y'(T/2)=0. \tag{2.51}$$

Multiplying both parts (2.50) on $y(t)$, after integration on the segment [0,T/2] with the account (2.51) we shall obtain

$$\int_0^{T/2} y'^2(s)ds = \frac{\rho}{\sqrt[\alpha]{2}}\left(\int_0^{T/2} y^{\frac{2\alpha}{\alpha-1}}(s)ds\right)^{1-\frac{1}{\alpha}}. \qquad (2.52)$$

Now multiplying both parts (2.50) on $y'(t)$, after integration on the segment [0,t] we have

$$y'^2(t) = 1 - \frac{\rho(\alpha-1)y^{\frac{2\alpha}{\alpha-1}}(t)}{\alpha\left(2\int_0^{T/2} y^{\frac{2\alpha}{\alpha-1}}(s)ds\right)^{1/\alpha}}. \qquad (2.53)$$

Integrating equality (2.53) on the segment [0,T/2] , with account (2.52) we find, that

$$\left(2\int_0^{T/2} y^{\frac{2\alpha}{\alpha-1}}(s)ds\right)^{1/\alpha} = \left[\frac{T\alpha}{(2\alpha-1)\rho}\right]^{\frac{1}{\alpha-1}} \qquad (2.54)$$

Recording (2.54) of (2.53) at $t \in [0, T/2]$ we obtain so that

$$y(0) = 0, \quad y(\frac{T}{2}) = \left[\frac{T\alpha^{\alpha}}{(2\alpha-1)(\alpha-1)^{\alpha-1}\rho^{\alpha}}\right]^{\frac{1}{2\alpha}} \qquad (2.55)$$

Having put in (2.55) $y(t) = y(T/2)z(t)$, we have:

$$dt = y(T/2)\frac{dz(t)}{\sqrt{1-z^{\frac{2\alpha}{\alpha-1}}(t)}}, \quad z(0) = 0, \quad z(T/2) = 1. \qquad (2.56)$$

Integrating (2.56) on the segment [0,T/2], we get

$$\frac{T}{2} = y(T/2)\int_0^1 \frac{ds}{\sqrt{1-s^{\frac{2\alpha}{\alpha-1}}}},$$

that confirms (2.49).

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