ON THE SOLUTIONS OF THIRD ORDER NON – LINEAR BOUNDARY VALUE PROBLEMS

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Мақолада учинчи тартибли икки нуқтавий чегаравий масалалар ечимининг мавжудлиги ва ягоналиги ҳақидаги теорема Грин функцияси ва унинг ҳосилаларини баҳолаш орқали келтириб чиқарилган.

В данной работе выведены условия существования и единственности с помощью оценки функции Грина и ее производных,

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Existence and uniqueness of solutions of third order non – linear boundary value problems are discussed. Two different methods have been used, (i) contraction mopping principle and (ii) successive approximations. It is found that the results obtained by method (ii) are better than those given by (i) and by [1], [2].

1. Introduction

In this poper we are concerned with existence and uniqueness results for non-linear third order differential equation

(1.1) $y''' = f(x, y, y', y'')$

When the boundary conditions are defined at two points. It will always be assumed that the function $f(x, y, y', y'')$ is continuous in the interior of its domain.

The poper is organized as follows. In Section2 we give auxiliary lemmas which will be employed heavily in the sequel are given. In Section3, some existence theorems are given. We also prove uniqueness of solutions by using contraction mopping. In Section4, by using successive approximation R. P. Agarwal's method [1], [2].

2. Auxiliary lemmas
3.

In this section we collect auxiliary lemmas which will be used later.

Let

$$(2.1)\quad G_1(x,t) = -\frac{1}{2}\begin{cases}\left[\dfrac{(x-a)(b-t)}{b-a}\right]^2 - (x-t)^2, & a \le t \le x,\\ \left[\dfrac{(x-a)(b-t)}{b-a}\right]^2, & x \le t \le b,\end{cases}$$

$$(2.2)\quad G_2(x,t) = -\frac{1}{2}.\begin{cases}(t-a)(b-t) - \dfrac{(b-x)^2(t-a)}{b-a}, & a \le t \le x,\\ \dfrac{(x-a)^2(b-t)}{b-a}, & x \le t \le b;\end{cases}$$

and

$$(2.3)\quad G_3(x,t) = -\begin{cases}(x-a)(t-a) - \dfrac{(t-a)^2}{2}, & a \le t \le x,\\ \dfrac{(x-a)^2}{2}, & x \le t \le b\end{cases}$$

are Green functions these boundary value problems:

(2.4) $y''' = 0,\quad y(a) = 0 \quad y'(a) = 0,\quad y(b) = 0,$

(2.5) $y''' = 0, \quad y(a) = 0, \quad y'(a) = 0, \quad y'(b) = 0$,

and

(2.6) $y''' = 0, \quad y(a) = 0, \quad y'(a) = 0, \quad y''(b) = 0$

respectively.

Lemma 2.1. Let $y(x) \in C^3[a,b]$ satisfying (2.4). Then

$$\left|y^{(k)}(x)\right| \le C_{1,k}(b-a)^{3-k}. \qquad Q, \qquad k = 0,1,2,$$

where $Q = \max_{a \le x \le b} |y'''(x)|$, and

$$C_{1,0} = \frac{2}{81}, \quad C_{1,1} = 1, \qquad C_{1,2} = \frac{2}{3}.$$

Proof. Any function $y(x) \in C^3[a,b]$ satisfying (2.4) can be written as

$$y(x) = \int_a^b G_1(x,t) y'''(t) dt,$$

where $G_1(t,s)$ is the Green's function of the boundary value problem (2,4), that is (2,1).

Thus we need to show that

$$\int_a^b \left|\frac{\partial^k G_1(x,t)}{\partial x^k}\right| dt \le C_{1,k}(b-a)^{3-k}, \quad k = 0,1,2$$

Since

$$\left|G_1(x,t)\right| = \frac{1}{2}\begin{cases}\left[\frac{(x-a)(b-t)}{b-a}\right]^2 - (x-t)^2, & a \le t \le x,\\ \left[\frac{(x-a)(b-t)}{b-t}\right]^2, & x \le t \le b,\end{cases}$$

then

$$\int_a^b \left|G_1(x,t)\right| dt = \frac{(x-a)^2(b-x)}{6} \equiv A_1(x) \le \frac{2}{81}(b-a)^3,$$

and (2.7) holds for k=0.

Since $\left|\frac{\partial G_1(x,t)}{\partial x}\right| = \frac{x-a}{(b-a)^2} \cdot \begin{cases}(t-a)\left|t-t^x\right|, & a \le t \le x,\\ (b-t)^2, & x \le t \le b,\end{cases}$

where

$$b - t^x = \frac{(b-a)(b-x)}{x-a}, \quad t^x = t(x),$$

Then we have

$$\int_a^b \left|\frac{\partial G_1(x,t)}{\partial x}\right| dt = \frac{(x-a)^2}{6} f(\eta),$$

where $f(\eta) = 2\eta - 3 + 2\eta(2-\eta)^3, \qquad \eta = \frac{b-a}{x-a} \triangleright 1.$

From

$$f^{1}(\eta)=0 \lhd\!=\!\rhd \quad \eta^{3}-\frac{9}{2}\eta^{2}+6\eta-\frac{9}{4}=0 \qquad \text{on}$$

$$\left(\eta-\frac{3}{2}\right)^{3}-\frac{3}{4}\left(\eta-\frac{3}{2}\right)=0;\quad \eta_{1}=3/2,\qquad \eta_{2}=\frac{3+\sqrt{3}}{2}\rhd 2$$

$f''(\eta_{1})\rhd 0,$ where $f''(\eta)=24(\eta-1)(2-\eta)$

Then $f(\eta)\le \quad (f(1),f(2))=1$

$$\int_{a}^{b}\left|\frac{\partial G_{1}(x,t)}{\partial x}\right|dt\le\frac{(x-a)}{G}\equiv B(x)$$

and (2.7) holds for k=1.

Since $\left|\frac{\partial^{2}G_{1}(x,t)}{\partial x^{2}}\right|=\frac{1}{(b-a)^{2}}\cdot\begin{cases}(b-a)^{2}-(b-t)^{2}, & a\le t\le x,\\ (b-t)^{2}, & x\le t\le b\end{cases}$

Then we have

$$\int_{a}^{b}\left|\frac{\partial G_{1}(x,t)}{\partial x}\right|U\le\frac{(x-a)^{2}}{4}.$$

Since $\left|\frac{\partial^{2}G_{1}(x,t)}{\partial x^{2}}\right|=\frac{1}{(ba)^{2}}\cdot\begin{cases}(b-a)^{2}-(b-t)^{2}, & a\le t\le x\\ (b-t)^{2}, & x\le t\le b\end{cases}$

Then we have

$$\int_{a}^{b}\left|\frac{\partial^{2}G_{1}(x,t)}{\partial x^{2}}\right|dt=\frac{2}{3}(b-a)-(b-x)+\frac{2}{3}\cdot\frac{(f-x)^{3}}{(f-a)^{2}},\qquad a\le x\le b.$$

We set $b-a=\eta(b-x).$ Then

$$\int_{a}^{b}\left|\frac{\partial^{2}G_{1}(x,t)}{\partial x^{2}}\right|dt=(b-x)g(\eta)=\frac{b-x}{3}g(\eta),$$

where

and $g(\eta)=\frac{2\eta^{3}-3\eta^{2}+2}{\eta},\quad \eta\ge 1$

$$g''(\eta)=\frac{4}{\eta^{4}}\rhd 0,\quad \eta\ge 1,\quad \eta\ge 1$$

then $g(\eta)\le 1?$ *and*

$$\int_{a}^{b}\left|\frac{\partial^{2}G_{1}(x,t)}{\partial x^{2}}\right|dt\le\frac{bx}{3}\equiv C(x)\le\frac{3x}{3}$$

$$\int_{a}^{b}\left|\frac{\partial^{2}G_{1}(x,t)}{\partial x^{2}}\right|dt\le\begin{cases}\frac{2}{3}(b-a), & eee\ \ a\le t\le\frac{ba}{\sqrt{2}}\\ \frac{1}{2}(ba) & eee\quad \frac{ba}{\sqrt{2}}\le t\le f\end{cases}$$

$$\int_{a}^{b}\left|\frac{\partial^{2}G_{1}(x,t)}{\partial x^{2}}\right|U\le\frac{2}{3}(ba),\quad a\le t\le f$$

Lemma 2.2. Let $y(x)\in C^{3}[a,b]$ satisfying (2.4) and

$$A_{1}(x)=\frac{(x-a)^{2}(b-x)}{6},\ B_{1}(x)=\frac{(x-a)^{2}}{4};\quad C_{1}(x)=\frac{2(b-a)}{3}.$$

Then

$$\int_a^b |G_1(x,t)| A_1(t)dt \le \frac{(b-a)^3}{60} A_1(x), \quad \int_a^b |G_1(x,t)| B_1(t)dt \le \frac{3(b-a)^2}{40} A_1(x),$$

$$\int_a^b |G_1(x,t)| C_1(t)dt \le \frac{b-a}{4} A_1(x), \quad a \le x \le b;$$

$$\int_a^b \left|\frac{\partial G_1(x_1 t)}{\partial x}\right| A_1(t)dt \le \frac{(b-a)^3}{90} B_1(x); \quad \int_a^b \left|\frac{\partial G_1(x,t)}{\partial x}\right| B_1(t) \le \frac{11}{120}(ba)^2 B_1(x),$$

$$\int_a^b \left|\frac{\partial G_1(x,t)}{\partial x}\right| C_1(t)dt \le \frac{(b-a)}{2} B_2(x), \quad a \le x \le b;$$

$$\int_a^b \left|\frac{\partial G_1(x,t)}{\partial x^2}\right| A_1(t)dt \le \frac{(b-a)^3}{45} C_1(x), \quad \int_a^b \left|\frac{\partial^2 G_1(x,t)}{\partial x^2}\right| B_1(t)df \le \frac{3(b-a)^2}{20} C_1(x);$$

$$\int_a^b \left|\frac{\partial^2 G_1(x,t)}{\partial x^2}\right| C_1(t)dt \le \frac{(b-a)}{4} C_1(x), \quad a \le x \le b.$$

The proof of the lemma2.2 involves computation which is tedious though elementary. In fact, $\frac{\partial G_1(x,t)}{\partial x}$ as well as $\frac{\partial^2 G_1(x,t)}{\partial x^2}$ both chagea sign and hence several different cases have to be considered separately, whereas $\frac{\partial^2 G_1(x,t)}{\partial x^2}$ has a chage of sign along the diagonal. The proof we omitted.

Let $G_2(x,t)$ *and* $G_3(x,t)$ are Green functboundary value problems (2.5) and (2.6) respectivally. Then we may prove these lemms.

Lemma 2.3 Let $y(t) \in C^3[a,b]$ satisfying (2.5). Then

$$\left|y^{(k)}(x)\right| \le C_{2,k}(b-a)^{3-k} Q, \; k = 0,1,2,$$

Where

$Q = \max_{a \le x \le b} |y'''(x)|$

And

$$C_{2,0} = \frac{1}{12}, \qquad C_{2,1} = \frac{1}{8}, \qquad C_{2,2} = \frac{1}{2}.$$

Lemma 2.4. Let $y(t) \in C^3[a,b]$ satisfying (2.5) and

$$A_2(x) = \frac{(x-a)^2(3b-a-2x)}{12}, \quad B_2(x) = \frac{(b-x)(x-a)}{2}, \quad C_2(x) = \frac{(x-a)^2 + (b-x)^2}{2(b-a)}.$$

Then

$$\int_a^b |G_2(x,t)| A_2(t)dt \le \frac{5(b-a)^3}{10} A_2(x), \; \int |G_2(x,t)| B_2(t) \le \frac{\sqrt{5}+1}{32}(b-a)^2 A_2(x),$$

$$\int |G_2(x,t)| C_2(t)dt \le \frac{3(b-a)}{10} A_2(x), \; a \le x \le b,$$

$$\int_a^b \left|\frac{\partial G_2(x,t)}{\partial x}\right| A_2(t)dt \le \frac{7(b-a)^3}{120} B_2(x); \quad \int \left|\frac{\partial G_2(x,t)}{\partial x}\right| B_2(t)dt \le \frac{5}{48}(b-a)^2 B_2(x),$$

$$\int_a^b \left|\frac{\partial G_2(x,t)}{\partial x}\right| C_2(t)dt \le \frac{b-a}{3} B_2(x),\ a \le x \le b;$$

$$\int_a^b \left|\frac{\partial^2 G_2(x,t)}{\partial x^2}\right| A_2(t)dt \le \frac{(b-a)^3}{120} C_2(x),\ \int_a^b \left|\frac{\partial^2 G_2(x,t)}{\partial x^2}\right| B_2(t)dt \le \frac{(b-a)^2}{6} C_2(x),$$

$$\int \left|\frac{\partial^2 G_2(x,t)}{\partial x^2}\right| C_2(t)dt \le \frac{7(b-a)}{24} C_2(x),\ a \le x \le b.$$

Lemma 2.5 Let $y(t) \in C^3[a,b]$ satisfying (2.6)

$$\left|y^{(k)}(x)\right| \le C_{3,k}(b-a)^{3-k} Q,\ k = 0,1,2,$$

Where

$$Q = \max_{a \le x \le b} \left|y'''(x)\right|$$

And

$$C_{3,0} = \frac{1}{3},\ C_{3,1} = \frac{1}{2},\ C_{3,2} = 1.$$

Lemma 2.6 Let $y(t) \in C^3[a,b]$ satisfying (2.6) and

$$A_3(x) = \frac{(x-a)^2(b-a)}{2} - \frac{(x-a)^3}{6};\ B_3(x) = (b-a)(x-a) - \frac{(x-a)^2}{2};\ C_3(x) = b-x.$$

Then

$$\int_a^b |G_3(x,t)| A_3(t)dt \le \frac{(b-a)^3}{6} A_3(x),\ \int |G_3(x,t)| B_3(t)dt \le \frac{2(b-a)^2}{5} A_3(x),$$

$$\int_a^b |G_3(x,t)| C_3(t)dt \le \frac{3(b-a)}{8} A_3(x),\ a \le x \le b;$$

$$\int_a^b \left|\frac{\partial G_3(x,t)}{\partial x}\right| A_3(t)dt \le \frac{7(b-a)^3}{b-a} B_3(x),\ \int \left|\frac{\partial G_3(x,t)}{\partial x}\right| B_3(t)dt \le \frac{5(b-a)^2}{12} B_3(x),$$

$$\int_a^b \left|\frac{\partial G_3(x,t)}{\partial x}\right| C_3(t)dt \le \frac{b-a}{2} B_3(x),\ a \le x \le b;$$

$$\int_a^b \left|\frac{\partial^2 G_3(x,t)}{\partial x^2}\right| A_3(t)dt \le \frac{(b-a)^3}{3} C_3(x),\ \int_a^b \left|\frac{\partial^2 G_3(x,t)}{\partial x^2}\right| B_3(t)dt \le \frac{(b-a)^2}{6} B_3(x),$$

$$\int_a^b \left|\frac{\partial^2 G_3(x,t)}{\partial x^2}\right| C_3(t)dt \le \frac{b-a}{2} C_3(x),\ a \le x \le b.$$

The proofs of the lemms 2.3—2.6 are similar of the proof lemms 2.1 and 2.2. because we omitted.

3. EXISTENCE AND UNIQUENESS

In this section, we prove a number of theorems, for the existence and uniqueness of solutions of (1.1).

Theorem.3.1. For some positive real numbers M, N, and L consider the set

$$\Omega = \left\{(x,y,u,v) : a \le x \le b, |y| \le 2M, |u| \le 2N, |v| \le 2L\right\}.$$

Let $|f| \le Q$ on Ω. If

$$b-a \le \min\left\{\frac{M-|A|-|C-A|}{2|B|}, \frac{2|C-A|}{N-3|B|}; \left(\frac{81M}{2Q}\right)^{\frac{1}{3}}, \left(\frac{6N}{Q}\right)^{\frac{1}{2}}, \frac{3L}{Q}\right\}$$

and $b-a \geq \dfrac{|B|+\sqrt{B^2+8L|C-A|}}{2L}$,

Then the boundary value problem

$$y''' = f(x, y, y', y''),$$
$$y(a) = A, y'(a) = B,\ y(b) = C$$

has a solution on [a,b].

Proof. The set

$$B[a,b] = \left\{y(x) \in C^2[a,b] : \|y_0\| \leq 2N, \|y_0'\| \leq 2N, \|y''_0\| \leq 2L\right\}$$

is a closed convex subset of the Banach space $C^2[a,b]$. The mapping T: $C^2[a,b] \to C^2[a,b]$ defined by

$$(3.4) \qquad (Ty)(x) = A + B(x-a) + [C - A - B(b-a)]\frac{(x-a)^2}{(b-a)^2} +$$

$$+ \int G_1(x,t) f(t, y(t), y'(t), y''(t))dt,$$

where G(x,t) is the Gree's function for the boundary value problem (2.4) is completely continuous. For $y \in B[a,b]$ we havely using (3.1), (3.2) and lemma2.1,

$$|(Ty)(x)| \leq |A| + 2|B|(b-a) + |C-A| + \frac{2(b-a)^3}{81}Q \leq 2M,$$
$$|(Ty)'(x)| \leq \frac{3|B|(b-a) + 2|C-A|}{b-a} + \frac{(b-a)^2}{6}Q \leq 2N,$$
$$|(Ty)''(x)| \leq \frac{|B|(b-a) + 2|C-A|}{(b-a)^2} + \frac{1}{3}(b-a)Q \leq 2L$$

This means T maps B[a,b] into itself. It then follows from the Schouder fixed—point theorem that T has a fixed point in B[a,b] wich is a solution of (3.3). Proof is complete.

Theorem 3.2.

For same positive real numbers M,N, and L consider the set $\Omega = \left\{(x,y,u,v) : a \leq x \leq b, |y| \leq 2M, |u| \leq 2N, |v| \leq 2L\right\}$. Let $|f| \leq Q$ *on* Ω.. If

$$b-a \leq \min\left\{\frac{2M-2|A|}{2|B|+|B-C|}, \left(\frac{12M}{Q}\right)^{\frac{1}{3}}, \left(\frac{8N}{Q}\right)^{\frac{1}{2}}, \left(\frac{2L}{Q}\right)\right\},$$

and

$$|B| + |B-C| \leq N,\ |B-C| \leq L(b-a),$$

then the boundary value problem

$$(3.5) \qquad y''' = f(x,y,y',y''),\quad y(a) = A,\quad y'(a) = B,\quad y'(b) = C$$

has solution on [a,b].

Theorem 3.3.

For somepositive real numbers M,N, and L consider the set $\Omega = \left\{(x,y,u,v) : a \leq x \leq b, |y| \leq 2M, |u| \leq 2N, |v| \leq 2L\right\}$. If

$$b-a \leq \min\left\{\frac{\sqrt{B^2 + 2|C|(M-|A|)} - |B|}{|C|}, \left(\frac{3M}{Q}\right)^{\frac{1}{3}}, \left(\frac{2N}{Q}\right)^{\frac{1}{2}}, \frac{L}{Q}\right\},$$

and

$$|B| + |C|(b-a) \leq N,\ |C| \leq: L,$$

then the foundary value problem

$$y''' = f(x, y, y', y''), \quad y(a) = A, \quad y'(a) = B, \quad y'(b) = C$$

has a solution on [a,b].

the proof of the theorems 3.2 and 3.3 is an analogously of the proof theorem 3.1 and omitted.

Now, let us assume that $f(x, y, y', y'')$ is Lipschitzian. Then we obtain.

Theorem 3.4

Let $f(x, y, y', y'')$ be continuous and

$$\left|f(x, y, y', y'') - f(x, \bar{y}, \bar{y}', \bar{y}'')\right| \le M_1\left|y - \bar{y}\right| + M_2\left|y' - \bar{y}'\right| + M_3\left|y'' - \bar{y}''\right|,$$

for $(x, y, u, v), (x, \bar{y}, \bar{u}, \bar{v}) \in [a,b] \times R^3$.

Then, of $\alpha_0 = \dfrac{2M_1(b-a)^3}{81} + \dfrac{M_2(b-a)^2}{6} + \dfrac{M_3(b-a)}{3} < 1,$

The boundary value problem (3.3) has a unique solution.

Proof. Let the space S consist of continuously differentiable functions on [a,b] with the norm

$$\|y\| = \max\left[M_1|y(x)| + M_2|y'(x)| + M_3|y''(x)|\right].$$

Define the operator T: $C^2[a,b] \to C^2[a,b]$ by (3.4). We shall show that T is contracting operator. For y(x) and $z(x) \in S$ we have, using (3.4), and lemma 2.1,

$$\left|(Ty)(x) - (Tz)(x)\right| \le \int |G_1(x,t)|\left[M_1|y(t) - z(t)| + M_2|y'(t) - z'(t)| + M_3|y''(t) - z''(t)|\right]dt \le$$

$$\|y - z\|\int |G_1(x,t)|dt \le \frac{2(b-a)^3}{81}\|y - z\|, \text{ i.e.}$$

$$\left|(Ty)(x) - (Tz)(x)\right| \le \frac{2(b-a)^3}{81}\|y - z\|.$$

$$\left|(Ty)'(x) - (Tz)'(x)\right| \le \|y - z\|\int \left|\frac{\partial G_1(x,t)}{\partial x}\right|dt \le \frac{b-a}{3}\|y - z\|$$

and

$$\left|(Ty)'(x) - (Tz)'(x)\right| \le \|y - z\|\int \left|\frac{\partial G_1(x,t)}{\partial x}\right|dt \le \frac{b-a}{3}\|y - z\|.$$

Therefore

$$\|Ty - Tz\| \le \left[\frac{2M_1(b-a)^3}{81} + \frac{M_2(b-a)^2}{6} + \frac{M_3(b-a)}{3}\right]\|y - z\| < \|y - z\|.$$

This inequality and contraction mapping principle yield the theorem.

By using lemms2.3 and 2.5 may be proves these theorems.

Theorem 3.5

[4] Let $f(x, y, y', y'')$ be continuous and satisfying (3.7). Then, if

$$\frac{M_1(b-a)^3}{12} + \frac{M_2(b-a)^2}{8} + \frac{M_3(b-a)}{2} < 1.$$

the boundary value problem (3.5) has a unique solution.

Theprem 3.6

[4] Let $f(x, y, y', y'')$ be continuous and satisfying (3.7). Then, if

$$\frac{M_1(b-a)^3}{3}+\frac{M_2(b-a)^2}{2}+M_3(b-a)<1.$$

the boundary value problem (3.6) has a unique solution.

The proof of the theorems 3.5 and 3.6 is on analogously of the proof theorem 3.4 and omitted.

4. AN IMPROVED BOUND FOR THE EXISTENCE AN UNIQUENESS 5.

In section3, we employed constraction mopping principle and we obtained conditions on the length of interval [a,b] in terms of Lipschitz contants and . Here we shall use the successive approximations M_1, M_2 method given $M_3>0$ by Ravi P.Agarwal [1], to find the length of the interval. The result obtained indicated that the solution exists over a lengs of interval greater than that jbtained in section3, and also greater than that obtained in [1], [2].

Theorem 4.1. Let $f(x,y,y',y'')$ be continuous and Lipschitzian on $[a,b]\times R^3$ and, where

$$\alpha_1=\frac{M_1(b-a)^3}{60}+\frac{3M_2(b-a)^2}{40}+\frac{M_3(b-a)}{4};$$
$$\alpha_2=\frac{M_1(b-a)^3}{90}+\frac{11M_2(b-a)^2}{120}+\frac{M_3(b-a)}{2};$$
$$\alpha_3=\frac{M_1(b-a)^3}{45}+\frac{3M_2(b-a)^2}{20}+\frac{M_3(b-a)}{4};$$

If, $\alpha<1$ then the boundary value problem (3.3) has a unique solution.

Proof. Define the successive oproximations

Then we have $$U_0(x)=P(x)=A+\frac{(x-a)(b-x)}{b-a}B+\frac{(x-a)^2}{(b-a)^2}(C-A)$$

$$U_{n+1}(x)=P(x)+\int_a^b G_1(x,t)f(t,u_n(t),u_n'(t),u_n''(t))dt$$

$$(4,1_0)\qquad |u_1(x)-u_0(x)|\le\frac{1}{6}Q(x-a)^2(b-x)=QA_1(x),\ a\le x\le b$$

$$(4,1_1)\qquad |u_1(x)-u_0(x)|\le\frac{1}{4}Q(x-a)^2=QB_1(x),\quad a\le x\le b$$

$$(4,1_2)\qquad |u_1''(x)-u_0''(x)|\le\frac{1}{2}Q(x-a)=QC_1(x),\ a\le x\le b.$$

Now, using the Lipschitz condition we have

$$|u(x)-u(x)|\le\int_a^b|G_1(x,t)|\left[M_1|u_1(t)-u_0(t)|+M_2|u_1'(t)-u_0'(t)|+M_3|u_1''(t)-u_0''(t)|\right]dt\le$$

$$\le Q\left[M_1\int_a^b|G_1(x,t)|A_1(t)dt+M_2\int_a^b|G_1(x,t)|B_1(t)dt+M_3\int_a^b|G_1(x,t)|C_1(t)dt\right]\le$$

$$\le Q\left[\frac{M_1(b-a)^3}{60}+\frac{3M_2(b-a)^2}{40}+\frac{M_3(b-a)}{4}\right]A_1(x)=\alpha_1QA_1(x).$$

There fore,

An analogously $$(4,2_0)\ |u_2(x)-u_1(x)|\le\alpha_1QA_1(x)\le\alpha QA_1(x),\qquad a\le x\le b,$$

$$(4,2_1) \qquad |u_2'(x)-u_1'(x)| \le \alpha_2 QB_1(x) \le \alpha QB_1(x), \ \ a \le x \le b,$$

and

$$(4,2_2) \qquad |u_2''(x)-u_1''(x)| \le \alpha_3 QC_1(x) \le \alpha QC_1(x), \ \ a \le x \le b.$$

By induction and using the above estimates

$$(4,n_0) \qquad |u_{n+1}(x)-u_n(x)| \le \alpha^n QA_1(x), \ \ a \le x \le b,$$
$$(4,n_1) \qquad |u_{n+1}(x)-u_n(x)| \le \alpha^n QB_1(x), \ \ a \le x \le b,$$
$$(4,n_2) \qquad |u_{n+1}(x)-u_n(x)| \le \alpha^n QC_1(x), \ \ a \le x \le b.$$

Since $\alpha < 1,$ estimates $(4,n_0)-(4,n_2)$ ensure that the boundary value problem (3.3) has a solution.

Now , we shall prove that the solution is inique, or in other words the boundary value problem (3.3) has a unique solution. Let us assume that there are two solutions, U(x) and V(x), and we defined

$$(4.3) \qquad \|u-v\| \le \max_{a \le x \le b}\left[M_1|u(x)-v(x)| + M_2|u'(x)-v'(x)| + M_3|u''(x)-v''(x)|\right].$$

Then we have

$$|u(x)-v(x)| \le A_1(x)\|u-v\|, \ a \le x \le b,$$
$$|u'(x)-v'(x)| \le B_1(x)\|u-v\|, \ a \le x \le b,$$
$$|u''(x)-v''(x)| \le C_1(x)\|u-v\|, \ \ a \le x \le b.$$

From (4.3) and above results, we find

$$|u(x)-v(x)| \le A_1(x)\|u-v\|, \ a \le x \le b,$$
$$|u'(x)-v'(x)| \le B_1(x)\|u-v\|, \ a \le x \le b,$$
$$|u''(x)-v''(x)| \le C_1(x)\|u-v\|, \ \ a \le x \le b.$$

Continuing this way, we have

$$|u(x)-v(x)| \le A_1(x)\|u-v\|, \ a \le x \le b,$$
$$|u'(x)-v'(x)| \le B_1(x)\|u-v\|, \ a \le x \le b,$$
$$|u''(x)-v''(x)| \le C_1(x)\|u-v\|, \ \ a \le x \le b.$$

Since $\alpha < 1$ result follows immediately. Proof is complete.
By using lemms 2.4 and 2.6, we may prove theirs theorems.

Theorem 4.2

Let $f(x,y,y',y'')$ be continuous and Lipschitzian on $[a,b]\times R^3$ and $\beta = \max(\beta_1,\beta_2,\beta_3)$ where

$$\beta_1 = \frac{M_1(b-a)^3}{24} + \frac{\sqrt{5}+1}{32}M_2(b-a)^2 + \frac{3}{10}M_3(b-a),$$
$$\beta_2 = \frac{7M_1(b-a)^3}{120} + \frac{5}{48}M_2(b-a)^2 + \frac{1}{3}M_3(b-a),$$
$$\beta_3 = \frac{M_1(b-a)^3}{120} + \frac{M_2(b-a)^2}{6} + \frac{7M_3(b-a)}{24}.$$

Then if $\beta < 1,$ the boundary value problem (3.4) has unique solution.

Theorem 4.3

Let $f(x,y,y',y'')$ be continuous and Lipschitzian on $[a,b]\times R^3$, $\gamma=\max(\gamma_1,\gamma_2,\gamma_3)$ where

$$\gamma_1=\frac{M_1(b-a)^3}{6}+\frac{2M_2(b-a)^2}{5}+\frac{3M_3}{8}(b-a),$$

$$\gamma_2=\frac{7M_1(b-a)^3}{60}+\frac{5M_2(b-a)^2}{12}+\frac{M_3}{2}(b-a),$$

$$\gamma_3=\frac{M_1(b-a)^3}{3}+\frac{M_2(b-a)^2}{6}+\frac{M_3}{2}(b-a).$$

If $\gamma<1$, then the boundary value problem (3.5) has a unique solution. The proofs of the theorems 4.2 and 4.3 are an analogously of the proof theorem 4.1, and are omitted.

Remark.1. The condition 3.8 of Theorem 3.4 is better than the condition of [1], which is
And of [2], which is

$$\frac{2M_1}{81}(b-a)^3+\frac{M_2}{6}(b-a)^2+\frac{2}{3}M_3(b-a)<1,$$

$$\frac{M_1(b-a)^3}{60}+\frac{M_2(b-a)^2}{6}+\frac{2}{3}M_3(b-a)<1.$$

3.By using the method of [3], we may prove the theorem better then theorem of [3].

Theorem 4.4.

Let be $f(x,y,y',y'')$ continuous and Lipschitzian on $[a,b]\times R^3$

$$\frac{M_1(b-a)^3}{60}+\frac{M_2(b-a)^2}{6}+\frac{M_3(b-a)}{4}<1,$$

than the boundary value problem (3.3) has a unique solution.

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