INVOLYUTSIYA XOSSASIGA EGA BO`LGAN ODDIY DIFFERENSIAL TENGLAMALARGA OID MISOLLAR YECHISH

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Maqolada involyutsiya xossasiga ega bo`lgan oddiy differensial tenglamalar va bir jinsli bo`lmagan birinchi va ikkinchi tartibli differrensial tenglamalarni yechishga oid misollar qaralgan.

***Kalit so`zlar:*** *involyutsiya, involyutsiya qatnashgan oddiy differensial tenglamalar, bir jinsli bo`lmagan involyutsiya qatnashgan differensial tenglama.*

В статье рассмотрены примеры решения задач обыкновенных, а также неоднородно обыкновенных дифференциальных уравнений первого и второго порядка, обладающих свойством инволюции.

Ключевые слова: *инволюция, обыкновенные дифференциальные уравнения с инволюцией, неоднородные дифференциальные уравнения с инволюцией.*

The author of this article studies the examples of solving the simple differential equations of involusive peculiarities and the equations of first and second differential orders

Key words: *involutsive, ordinary differential equations, homogeneous ordinary differential equations*

Ma'lumki, agar $\alpha : R \to R$ akslantirishda $\alpha(\alpha(x)) = x$ tenglik bajarilsa bu akslantirishga involyutsiya deyiladi.

Differensial tenglamalar kursida ham involyutsiya xossasiga ega bo`lgan differensial tenglamalar uchraydi. Misol uchun [1.100] da Zilbershteyn tenglamasi deb nomlangan (Silberstein L. *Solution of the equation* $f'(x) = f\left(\frac{1}{x}\right)$. // Phil. Mag.7th Series, 30(1940), p.185-186.)

$$f'(x) = f\left(\frac{1}{x}\right)$$

ko`rinishdagi tenglamani yechish tavsiya qilingan, ammo yechilmagan bo`lib, bu tenglamaga $\alpha(x) = \frac{1}{x}$ akslantirish tatbiq etilsa, $f \circ f(x) = x$ ko`rinishdagi ikkinchi tartibli involyutsiya mavjudligini ko`rish qiyin emas. Boshqacha aytilganda, x ni ketma-ket ikki marta $\frac{1}{x}$ ga akslantirilsa, dastlabki tenglamaga qaytiladi. Involyutsiya xossasiga ega bo`lgan differensial tenglamalar orasida kechikkan argumentli $y'(x) = y(a - x)$ tenglamalar ham uchraydi, chunki x ni ketma-ket ikki marta $a - x$ ga akslantirish natijasida dastlabki tenglamaga qaytiladi.

Bunday ko`rinishdagi misollarni ko`plab keltirish mumkin. Masalan: $\alpha(x) = \sqrt[m]{1 - x^m}$ $m \in N$, $\alpha(x) = \frac{ax + b}{cx - a}$ funksiya va uning xususiy hollari bo`lgan $\alpha(x) = 1 - x, \alpha(x) = \frac{1}{x}, \alpha(x) = \frac{x}{x-1}, \alpha(x) = \frac{x+1}{x-1}$ kabi funksiyalar involyutsiya hosil qiladi.

Shuning uchun birinchi darajali involyutsiyaga ega bo'lgan birinchi tartibli oddiy diferensial tenglamalar umumiy holda

$$y'(x) = y(\alpha(x))$$

ko`rinishda berilishi mumkin. Agar x ni $\alpha(x)$ bilan almashtirsak

$$y'(\alpha(x)) = y(x)$$

ifodani va differensial tenglamani differensiallash bilan

$$y''(x) = \alpha'(x)\, y'(\alpha(x)) = \alpha'(x)\, y(x),$$

ya'ni ikkinchi tartibli

$$y''(x) - \alpha'(x) y(x) = 0$$

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tenglamani hosil qilamiz. [2] maqolada involyutsiya va uning xossalari, differensial tenglamalar involyuisiyasi haqida ko`plab ma'lumotlar keltirilgan va bir qator misollar yechilgan. Mazkur maqola yuqoridagi ishlarni amaliy jihatdan to`ldirishga qaratilgan.

Endi involyutsiya qatnashgan oddiy differensial tenglamalarga oid misollar yechishdan namunalar keltiramiz.

1-misol [1]. Quyidagi tenglamani yeching: $y'(x)=y\left(\frac{1}{x}\right)$

Yechilishi. Berilgan tenglamani yechish uchun $x=e^t$, $y(x)=g(t)$ almashtirish kiritamiz. U holda

$$y(x)=y(e^t)=g(t)$$

bo`lgani uchun

$$y\left(\frac{1}{x}\right)=y(e^{-t})=g(-t)$$

bo‘ladi va bundan tashqari $t=\ln x$ bo‘lgani uchun

$$y'(x)=g'(t)\cdot\frac{dt}{dx}=\frac{1}{x}g'(t)=e^{-t}g'(t)$$

bo‘lib, Zilbershteyn tenglamasi

$$g'(t)=e^t g(-t)$$

ko‘rinishni oladi. Hosil bo`lgan tenglamani yana bir bor differensiallash bilan

$$g''(t)=e^t g(-t)-e^t g'(-t)=e^t\cdot e^{-t}g'(t)-e^t\cdot e^{-t}g(t)$$

ya'ni

$$g''(t)-g'(t)+g(t)=0$$

o‘zgarmas koeffisiyentli differensial tenglamani hosil qilamiz. Bu tenglamaning xarakteristik tenglamasi

$$k^2-k+1=0$$

va uning ildizlari

$$k_{1,2}=\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$$

bo‘lgani uchun hosil qilingan oddiy differensial tenglamaning umumiy yechimi

bo‘ladi. Bu tenglamadan

$$g(t)=e^{\frac{t}{2}}\left[C_1\cos\frac{t\sqrt{3}}{2}+C_2\sin\frac{t\sqrt{3}}{2}\right]$$

$$g'(t)=\frac{1}{2}g(t)+\frac{\sqrt{3}}{2}e^{\frac{t}{2}}\left[-C_1\sin\frac{t\sqrt{3}}{2}+C_2\cos\frac{t\sqrt{3}}{2}\right]=$$

$$=e^{\frac{t}{2}}\left[\frac{C_1+\sqrt{3}C_2}{2}\cos\frac{t\sqrt{3}}{2}+\frac{C_2-\sqrt{3}C_1}{2}\sin\frac{t\sqrt{3}}{2}\right]$$

bo‘lgani uchun

$$e^{-t}g'(t)=g(-t)$$

tenglikka ko‘ra

$$e^{-\frac{t}{2}}\left[C_1\cos\frac{t\sqrt{3}}{2}-C_2\sin\frac{t\sqrt{3}}{2}\right]=e^{\frac{t}{2}}\left[\frac{C_1+\sqrt{3}C_2}{2}\cos\frac{t\sqrt{3}}{2}+\frac{C_2-\sqrt{3}C_1}{2}\sin\frac{t\sqrt{3}}{2}\right]$$

Bu tenglikdan $C_1 = \frac{C_1 + \sqrt{3}C_2}{2}$, $-C_2 = \frac{C_2 - \sqrt{3}C_1}{2}$, ya'ni $C_1 = \sqrt{3}C_2$ ekanligini aniqlaymiz.

Demak, $$g(t) = C_2 e^{\frac{t}{2}}\left[\sqrt{3}\cos\frac{t\sqrt{3}}{2} + \sin\frac{t\sqrt{3}}{2}\right] = 2C_2 e^{\frac{t}{2}}\left[\cos\frac{\pi}{6}\cos\frac{t\sqrt{3}}{2} + \sin\frac{\pi}{6}\sin\frac{t\sqrt{3}}{2}\right] =$$

$$= Ce^{\frac{t}{2}}\cos\left(\frac{\pi}{6} - \frac{t\sqrt{3}}{2}\right)$$

bu yerda $C = 2C_2$. Ammo $g(t) = y(x)$, $t = \ln x$ bo`lgani uchun berilgan tenglamaning yechimi ko'rinishga ega. $$y(x) = C\sqrt{x}\cos\left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\ln|x|\right) = C\sin\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\ln|x|\right)$$

Ushbu $y'(x) = y(1-x)$ tenglamani yeching.

Yechilishi. Berilgan tenglamada x ni $1-x$ bilan almashtirsak,

$$y'(1-x) = y(x)$$

tenglikni hosil qilamiz. Berilgan tenglamani differensiallash bilan esa

yoki $$y''(x) = -y'(1-x) = -y(x)$$

$$y''(x) + y(x) = 0$$

tenglamani hosil qilamiz. Bu tenglamaning umumiy yechimi $y(x) = C_1 cosx + C_2 \sin x$ ko`rinishga ega va bu yechimda x ni $1-x$ bilan almashtirish va differensiallash

$$y(1-x) = C_1\cos(1-x) + C_2\sin(1-x) \quad \text{va} \quad y'(x) = -C_1\sin x + C_2\cos x$$

tengliklarni beradi. Bu tengliklar va berilgan tenglamaga ko`ra

$$-C_1\sin x + C_2\cos x = C_1\cos(1-x) + C_2\sin(1-x)$$

tenglikni hosil qilamiz. Bir qator hisoblashlardan so`ng bu tenglikdan

$$\begin{cases} C_1\cos 1 + C_2\sin 1 = C_2, \\ C_1\sin 1 - C_2\cos 1 = -C_1 \end{cases}$$

sistemaga, undan esa $C_2 = \frac{1+\sin 1}{\cos 1}C_1$ munosabatga ega bo`lganimiz uchun berilgan tenglamaning umumiy yechimi

$$y(x) = C\left(\sin x + \cos(1-x)\right)$$

ko`rinishga ega bo`ladi.

3-misol. Quyidagi ikkinchi tartibli

$$y''(x) = y\left(\frac{1}{x}\right)$$

tenglamaning umumiy yechimini toping.

Yechilishi. Berilgan tenglamani x bo`yicha ketma-ket differensiallab

$$y'''(x) = -\frac{1}{x^2}y'\left(\frac{1}{x}\right), \quad y^{(IV)}(x) = \frac{2}{x^3}y'\left(\frac{1}{x}\right) + \frac{1}{x^4}y''\left(\frac{1}{x}\right)$$

tengliklarni hosil qilamiz.

Endi berilgan tenglamada $f : x \to \frac{1}{x}$ almashtirish bajarsak, berilgan tenglama

$$y''\left(\frac{1}{x}\right) = y(x)$$

ko`rinishni oladi. Bundan va yuqorida hosil qilingan ikki tengliklardan

$$x^4 y^{(IV)}(x) + 2x^3 y'''(x) - y(x) = 0$$

Eylerning oddiy differensial tenglamasini hosil qilamiz. Tenglamaning xarakteristik ko`phadi:

$$k(k-1)(k-2)(k-3)+2k(k-1)(k-2)-1$$

$$k_{1,2}=1\pm\sqrt{\frac{1+\sqrt{5}}{2}}=1\pm\frac{\sqrt{2+2\sqrt{5}}}{2},\ \ k_{3,4}=1\pm i\sqrt{\frac{-1+\sqrt{5}}{2}}=1\pm i\frac{\sqrt{-2+2\sqrt{5}}}{2}$$

ildizlarga ega bo`lgani uchun berilgan differensial tenglamaning umumiy yechimi:

$$y(x)=x\left[C_1ch\left(\frac{\sqrt{2+2\sqrt{5}}}{2}\ln|x|\right)+C_2sh\left(\frac{\sqrt{2+2\sqrt{5}}}{2}\ln|x|\right)\right]+$$

$$+x\left[C_3\cos\left(\frac{\sqrt{-2+2\sqrt{5}}}{2}\ln|x|\right)+C_4\sin\left(\frac{\sqrt{-2+2\sqrt{5}}}{2}\ln|x|\right)\right]$$

4-misol [2]. Ushbu $y'(x)=y\left(\frac{1}{x}\right)+\varphi(x),\ \ y(1)=y_0,\ y'(1)=y_0+\varphi(1)$

Koshi masalasini yeching.

Yechilishi. Berilgan tenglamada x ni $\frac{1}{x}$ bilan almashtirsak,

$$y'\left(\frac{1}{x}\right)=y(x)+\varphi\left(\frac{1}{x}\right)$$

tenglikni hosil qilamiz. Ikkinchi tomondan berilgan tenglamani differensiallash va soddalashtirish natijasida

$$x^2y''(x)+y(x)=x^2\cdot\varphi'(x)-\varphi\left(\frac{1}{x}\right)$$

oddiy differensial tenglama hosil bo`ladi . Uning bir jinsli qismi

$$y(x)=\sqrt{x}\cdot\left[C_1\cos\frac{\sqrt{3}\ln x}{2}+C_2\sin\frac{\sqrt{3}\ln x}{2}\right]$$

umumiy yechimga ega.

Endi berilgan tenglamaning xususiy yechimini topish uchun o`zgarmasni variatsiyalash usulini qo`llaymiz va xususiy yechimni $y(x)=\sqrt{x}\cdot\left[C_1(E)\cos\frac{\sqrt{3}\ln x}{2}+C_2(E)\sin\frac{\sqrt{3}\ln x}{2}\right]$

ko‘rinishda izlaymiz. Bu tenglikdan noma’lum

$$C_1(x)=C_{10}-\frac{2}{\sqrt{3}}\cdot\int_1^x t^{\frac{1}{2}}\sin\frac{\sqrt{3}\ln t}{2}\left(t^2\phi'(t)-\phi\left(\frac{1}{t}\right)\right)dt$$

$$C_2(x)=C_{20}-\frac{2}{\sqrt{3}}\cdot\int_1^x t^{\frac{1}{2}}cos\frac{\sqrt{3}\ln t}{2}\left(t^2\phi'(t)-\phi\left(\frac{1}{t}\right)\right)dt$$

koeffisiyentlarni aniqlab, tenglama yechimini ifodalovchi funksiyaga qoyish bilan bir qator soddalashtirgandan so‘ng umumiy yechim:

$$y(x)=C_{10}\sqrt{x}\cos\frac{\sqrt{3}\ln x}{2}+C_{20}\sqrt{x}\sin\frac{\sqrt{3}\ln x}{2}+\frac{2\sqrt{x}}{\sqrt{3}}\cdot\int_1^x t^{\frac{1}{2}}\left[t^2\phi'(t)-\phi\left(\frac{1}{t}\right)\right]\cdot\sin\left[\frac{\sqrt{3}}{2}\ln\left(\frac{x}{t}\right)\right]dt$$

ko‘rinishda bo`lishini, ikkinchi tomondan berilgan boshlang‘ich shartlarga asosan $C_{10}=y_0$ va $C_{20}=\frac{2}{\sqrt{3}}\cdot\left[\frac{1}{2}y_0+\phi(1)\right]$ ekanligini aniqlaymiz. Shuning uchun yuqorida berilgan Koshi masalasining yechimi :

$$y(x)=y_0\sqrt{x}\cos\frac{\sqrt{3}\ln x}{2}+\frac{2}{\sqrt{3}}\left[\frac{1}{2}y_0+\phi(1)\right]\sqrt{x}\sin\frac{\sqrt{3}\ln x}{2}+$$

$$+\frac{2\sqrt{x}}{\sqrt{3}}\cdot\int_1^x t^{\frac{1}{2}}\left[t^2\phi'(t)-\phi\left(\frac{1}{t}\right)\right]\cdot\sin\left[\frac{\sqrt{3}}{2}\ln\left(\frac{x}{t}\right)\right]dt$$

ko‘rinishga ega.

Biz mustaqil tadqiqot olib borishlari uchun talabalarga involyutsiya xossasiga ega bo‘lgan quyidagi differensial tenglamalarni yechishni tavsiya qilamiz:

1) $y'(x)=y(1-x)$; 2) ; $y'(x)=\frac{1}{y(a-x)}$ 3) $y'(x)=y\left(\frac{x}{x-1}\right)$; 4) $y'(x)=y\left(\frac{x+1}{x-1}\right)$;

5) $y'(x)=y\left(\sqrt{1-x^2}\right)$; 6) $y''(x)=\frac{1}{y(1-x)}$; 7) $y''(x)=y(1-x)+\varphi(x)$.

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