VIRTINGER TENGSIZLIGINING TURLI ISBOTLARI

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Maqolada Virtinger tengsizligining isbotlari keltirilgan va bu tengsizlikning izoperimetrik masalalarga tatbiqi ko'rib chiqilgan.

Kalit so'zlar: *Virtinger tengsizligi, izoperimetrik masala.*

В статье приведены доказательства неравенства Виртингера и рассмотрены применения этого неравенства к изопериметрическим задачам.

Ключевые слова: *неравенство Виртингера, изопериметрическая задача.*

The article proves the Wirtinger's inequality and considers application of this inequality to an isometrical tasks.

Key words: *Wirtinger's inequality, izometrical problem.*

Matematik analiz kursining eng mashhur tengsizliklaridan biri Virtinger tengsizligi bo'lib, u quyidagicha ifodalanadi:

Teorema. Agar davri 2π ga teng bo'lgan $x(t)$ funksiya uchun

$$\int_0^{2\pi} x(t)\,\mathrm{dt}=0 \tag{1}$$

tenglik bajarilsa, u holda

$$\int_0^{2\pi} x'^2(t)\,dt \geq \int_0^{2\pi} x^2(t)dt \tag{2}$$

tengsizlik o'rinli, bu yerda tenglik belgisi

$$x(t) = C_1 \sin t + C_2 \cos t \tag{3}$$

funksiya uchun bajariladi.

Bu yerda tengsizlikning turli xil isbotlari jamlangan bo'lib, asosiy adabiyotlar sifatida [1,2] qaraldi. Tengsizlikning turli isbotlaridan biri sifatida optimal boshqaruv nazariyasining L.S.Pontryagin maksimum prinsipiga asoslangan tengsizlikning to'rtinchi isboti ham berildi.

Teoremaning isboti.

1-usul. Haqiqiy $x(t)$ funksiyaning kompleks ko'rinishdagi

$$x(t) = \sum_{k=1}^{\infty} a_k e^{ikt} \tag{4}$$

Fur'e qatorini qaraymiz. U holda

$$x'(t) = \sum_{k=1}^{\infty} a_k ike^{ikt} \tag{5}$$

$x(t)$ va $x`(t)$ funksiyalar uchun

$$\int_0^{2\pi} x^2(t)\,dt = \sum_{k=1}^{\infty} |a_k|^2,\ \int_0^{2\pi} x'^2(t)\,dt = \sum_{k=1}^{n} n^2 |a_k|^2$$

munosabatlardan Parseval-Bessel tengsizligiga ko'ra

$$\int_0^{2\pi} x'^2(t)\,dt = \sum_{k=1}^{n} n^2 |a_k|^2 \sum_{k=1}^{\infty} |a_k|^2 \geq \int_0^{2\pi} x^2(t)\,dt$$

Bundan (2) tengsizlikning o'rinli ekani kelib chiqadi. Agarda hech bo'lmaganda k ning bitta qiymati uchun

$$a_n \neq 0, \quad |k| \geq 2$$

tengsizliklar bajarilsa, (2) tengsizlik qat'iy bajariladi.

2-usul. Bu usul variatsion hisobga asoslangan bo'lib, $p(t)$ chegaralangan musbat funksiya uchun

$$\int p(t)\,x\ \ (t)\,dt \tag{6}$$

shart bajarilganda

$$J(x) = \int_0^{2\pi} x'^2(t)\,dt$$

funksionalning $x(0) = x(2\pi) = 0$ chegaraviy shartlarni qanoatlantiruvchi minimal qiymatini topish talab qilinadi.

Lagranj ko'paytuvchisini qo'llab bu masalani

$$J(x) = \int_0^{2\pi} \left(x'^2(t) - \lambda p(t) x^2(t)\right) dt \tag{7}$$

funksionalning $x(0) = x(2\pi) = 0$ chegaraviy shartlarni qanoatlantiruvchi minimal qiymatini topish masalasiga keltirish mumkin.

(7) funksional uchun Eyler tenglamasi

$$x'' + \lambda p(t) x = 0 \tag{8}$$

ko‘rinishga ega. Agar $x(t)$ ekstremal (8) tenglamani qanoatlantirsa, u holda bo‘laklab integrallash bilan

$$\int_0^{2\pi} x\left(x'' + \lambda p(t)x\right)dt = xx' \Big|_0^{2\pi} - \int_0^{2\pi} x'^2(t)\,dt + \lambda\int_0^{2\pi} p(t)x^2 = 0$$

tenglikka va bundan

$$\int_0^{2\pi} x'^2(t)\,dt = \lambda$$

tenglikka kelamiz. Demak, topilishi talab qilingan minimal qiymat

$$x'' + \lambda p(t)x = 0,\ x(0) = x(2\pi)$$

Shturm-Liuvil masalasining eng kichik xos qiymati λ dan iborat ekan.

3-usul.

$$z(t) = x(t+\pi) - x(t)$$

funksiya $t = 0$ va $t = \pi$ nuqtalarda turli ishorali qiymatlar qabul qilgani uchun $0, \pi$ oraliqning qandaydir $t = \alpha$ nuqtasida nolga teng bo‘ladi.

Aytaylik, $z(\alpha) = 0, 0 \le \alpha < \pi$ va $x(\alpha) = a$ bo‘lsin.

$x' \in L^2$ bo‘lgani uchun, $(x-a)^2\, ctg(t-\alpha)$ funksiya $t = \alpha$ va $t = \alpha + \pi$ nuqtalarda nolga teng bo‘ladi.

Shuning uchun

$$\int_0^{2\pi}\left[x'^2 - (x-a)^2 - \left\{x' - (x-a)\,ctg(t-\alpha)\right\}^2\right]dt =$$

$$= \left[(x-a)^2\, ctg(t-\alpha)\right]_0^{2\pi} = 0.$$

Endi (1) munosabatni inobatga olgan holda quyidagi

$$\int_0^{2\pi}(x'^2 - x^2)\,dt = 2\pi a^2 + \int_0^{2\pi}\left[x' - (x-a)\,ctg(t-\alpha)\right]^2 dt$$

tenglikdan

$$\int_0^{2\pi} x'^2 dt \ge \int_0^{2\pi} x^2 dt$$

tengsizlikni hosil qilamiz. Bu tengsizlikda tenglik belgisi $a = 0$ va $x' = xctg(t-\infty),\ x(t) = C\sin(t-\infty)$ funksiya uchun o‘rinli ekani ravshan.

4-usul. Endi (2) tengsizlikning isbotini L.S.Pontryagin [3] maksimum prinsipiga asoslangan holda olib boramiz. Buning uchun quyidagi optimallashtirish masalasini qaraymiz:

$$\dot{x} = u,\ x(0) = x(2\pi),\ \int_0^{2\pi} x(t)\,dt = 0$$

Bu shartlar bajarilganda,

$$\int_0^{2\pi} u^2(t)\,dt \ge \int_0^{2\pi} x^2(t)\,dt$$

tengsizlik o‘rinli bo‘lishini ko‘rsatish uchun

$$\int_0^{2\pi}\left(u^2 - x^2 - \lambda x\right)dt \to inf$$

funksionalni qaraymiz.

Pontragin maksimumi prinsipiga ko‘ra

$$H = \psi u - \left(u^2 - x^2 - \lambda x\right)$$

funksiyani tuzamiz. Exstremumning zaruriy shartiga ko‘ra

$$\frac{\partial H}{\partial u} = 0 \Rightarrow \quad u = \frac{\psi}{2}$$

Qo‘shma sistema esa

$$\dot{\psi} = -\frac{\partial H}{\partial x} \Rightarrow \dot{\psi} = -2x - \lambda$$

ko‘rinishga ega bo‘lgani uchun

$$\ddot{x} = \dot{u} = \frac{\dot{\psi}}{2} = -\frac{2x + \lambda}{2} = -x - \frac{\lambda}{2},$$

yoki

$$\ddot{x} + x = -\frac{\lambda}{2}$$

o‘zgarmasni variatsiyalash usulini qo‘llab bu tenglamaning umumiy yechimi

$$x(t) = A\cos t + B\sin t - \frac{\lambda}{2}$$

ko‘rinishda ekanligini aniqlaymiz. Ammo

$$0 = \int_0^{2\pi} x(t)dt = C_{10}\int_0^{2\pi} \cos t + C_{20}\int_0^{2\pi} \sin t = -\frac{\lambda}{2}\cdot 2\pi = -\lambda$$

Bundan $\lambda = 0$ va shu bilan birga doimo $\int_0^{2\pi} \dot{x}^2(t)\,dt \geq 0$ ekanligini e’tiborga olinsa,

$$\int_0^{2\pi} \left(u^2 - x^2 - 0\cdot x\right)dt \geq 0 \Rightarrow \int_0^{2\pi} \dot{x}^2 dt \geq \int_0^{2\pi} x^2 dt$$

Bu tengsizlikda tenglik belgisi $x(t) = A\cos t + B\sin t$ funksiya uchun bajariladi.

5-usul. Virtinger tengsizligini variatsion hisob usuli bilan ham keltirib chiqarish mumkin. Buning uchiun

$$\frac{\int_0^{\pi} x'^2(t)\,dt}{\int_0^{\pi} x^2(t)\,dt} \to \min, \quad \int_0^{\pi} x(t)\,dt = 0$$

variatsion hisob masalasini qaraymiz. Bu masalaning yechimi mavjud bo‘lsin deb faraz qilamiz, ya’ni

$\frac{\int_0^{\pi} x'^2(t)\,dt}{\int_0^{\pi} x^2(t)\,dt} = \lambda$ bo‘lsin. U holda $\int_0^{\pi}\left[x'^2(t) - \lambda x^2(t) + x(t)\right]dt \to \min$

Endi $F = x'^2 - \lambda x^2 + x$ funksiya uchun $F_{x'} = 2x', F_x = -2\lambda x + 1$ bo‘lgani cababli variatsion hisobning $-\frac{d}{dt}F_{x'} + F_x = 0$ Eyler tenglamasiga asosan $x'' + \lambda x = 1$ tenglamaga ega bo‘lamiz. Bu tenglamaning umumiy yechimi

$$x(t) = C_1\cos\sqrt{\lambda}t + C_2\sin\sqrt{\lambda}t - \frac{1}{\lambda}$$

bo’lgani uchun kelgusi hisoblashlar 4-usuldagidek olib boriladi.

Virtinger tengsizligining izoperimetrik masalaga tatbiqi

(2)–Virtinger tengsizligi doiraning izoperimetrik xossasini isbotlashga asos bo‘ladi.

S yuzali chegaralovchi L uzunlikdagi sodda yopiq l chiziq qutb koordinatalari sistemasida

$$l : x = x(\varphi), y = y(\varphi) \quad 0 \leq \varphi \leq 2\pi$$

tenglama bilan berilgan bo‘lsin. l chiziq yoyining s uzunligi bo‘yicha

$$\varphi = \frac{2\pi s}{L}$$

parametr kiritamiz. Qulaylik uchun $x(\varphi)$ va $y(\varphi)$ funksiyalarni differensiallanuvchi funksiyalar deb qaraymiz. l yopiq chiziq bo‘lgani uchun

$$\int_0^{2\pi} y d\varphi = 0$$

U holda

$$\left(\frac{dx}{d\varphi}\right)^2 + \left(\frac{dy}{d\varphi}\right)^2 = \left\{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right\}\frac{L^2}{4\pi^2} = \frac{L^2}{4\pi^2},$$

chunki $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$

$$\frac{L^2}{2\pi} - 2S = \int_0^{2\pi}\left\{\left(\frac{dx}{d\varphi}\right)^2 + \left(\frac{dy}{d\varphi}\right)^2\right\}d\varphi + 2\int_0^{2\pi} y\frac{dx}{d\varphi}d\varphi =$$

$$= \int_0^{2\pi}\left(\frac{dx}{d\varphi} + y\right)^2 d\varphi + \int_0^{2\pi}\left\{\left(\frac{dy}{d\varphi}\right)^2 - y^2\right\}d\varphi \geq$$

$$\geq \int_0^{2\pi}\left\{\left(\frac{dy}{d\varphi}\right)^2 - y^2\right\}d\varphi \geq 0.$$

Shunig uchun

$$S \leq \frac{L^2}{4\pi}$$

izoperimetrik tengsizlik o‘rinli.

Bu yerda tenglik $y = A\cos\varphi + B\sin\varphi$ va $x = -\int y d\varphi = -A\sin\varphi + B\cos\varphi + C$ bo‘lganda, ya’ni $l: (x-C)^2 + y^2 = A^2 + B^2$ doira uchun bajariladi.

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