VIRTINGER TENGSIZLIGINING TURLI ISBOTLARI

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Maqolada Virtinger tengsizligining isbotlari keltirilgan va bu tengsizlikning izoperimetrik masalalarga tatbiqi qaralgan.

Kalit so‘zlar: *Virtinger tengsizligi, izoperimetrik masala.*

В статье приведены доказательства неравенства Виртингера и рассмотрены применения этого неравенства к изопериметрическим задачам.

Ключевые слова: неравенство Виртингера, изопериметрическая задача.

The article provides the proof of the Vertinger inequality and and considers its application in isometric problems.

Key words: *Wirtinger’s inequality, izometric problems.*

Matematik analiz kursining eng mashhur tengsizliklaridan biri Virtinger tengsizligi bo‘lib, u quyidagicha ifodalanadi:

Teorema. Agar davri 2π ga teng bo‘lgan $x(t)$ funksiya uchun

$$\int_0^{2\pi} x(t)\,\mathrm{dt}=0 \qquad (1)$$

tenglik bajarilsa, u holda

$$\int_0^{2\pi} x'^2(t)\,dt \geq \int_0^{2\pi} x^2(t)dt$$

tengsizlik o‘rinli, bu yerda tenglik belgisi

$$x(t) = C_1 \sin t + C_2 \cos t$$

funksiya uchun bajariladi.

Bu yerda tengsizlikning turli xil isbotlari jamlangan bo‘lib, asosiy adabiyotlar sifatida [1,2] qaraldi. Tengsizlikning turli isbotlaridan biri sifatida optimal boshqaruv nazariyasining L.S.Pontryagin maksimum prinsipiga asoslangan tengsizlikning to‘rtinchi isboti ham berildi.

Teoremaning isboti.

1-usul. Haqiqiy $x(t)$ funksiyaning kompleks ko‘rinishdagi

$$x(t) = \sum_{k=1}^{\infty} a_k e^{ikt} \qquad (4)$$

Fur’e qatorini qaraymiz. U holda

$$x'(t) = \sum_{k=1}^{\infty} a_k ike^{ikt} \qquad (5)$$

va $x'(t)$ funksiyalar uchun

$$\int_0^{2\pi} x^2(t)\,dt = \sum_{k=1}^{\infty} |a_k|^2, \quad \int_0^{2\pi} x'^2(t)\,dt = \sum_{k=1}^{n} n^2 |a_k|^2$$

munosabatlardan Parseval-Bessel tengsizligiga ko‘ra

$$\int_0^{2\pi} x'^2(t)\,dt = \sum_{k=1}^{n} n^2 |a_k|^2 \geq \sum_{k=1}^{\infty} |a_k|^2 \int_0^{2\pi} x^2(t)\,dt.$$

Bundan (2) tengsizlikning o‘rinli ekani kelib chiqadi. Agarda hech bo‘lmaganda ning bitta qiymati uchun

$$a_n \neq 0, \quad |k| \geq 2$$

tengsizliklar bajarilsa, (2) tengsizlik qat’iy bajariladi.

2-usul. Bu usul variatsion hisobga asoslangan bo‘lib, $p(t)$ chegaralangan musbat funksiya uchun

$$\int_0^{2\pi} p(t)x^2(t)dt = 1 \qquad (6)$$

shart bajarilganda

$$J(x) = \int_0^{2\pi} x'^2(t)dt$$

funksionalning $x(0) = x(2\pi) = 0$ chegaraviy shartlarni qanoatlantiruvchi minimal qiymatini topish talab qilinadi.

Lagranj ko‘paytuvchisini qo‘llab bu masalani

$$J(x) = \int_0^{2\pi} \left(x'^2(t) - \lambda p(t)x^2(t)\right)dt \qquad (7)$$

funksionalning $x(0) = x(2\pi) = 0$ chegaraviy shartlarni qanoatlantiruvchi minimal qiymatini topish masalasiga keltirish mumkin.

(7) funksional uchun Eyler tenglamasi

$$x'' + \lambda p(t)x = 0 \qquad (8)$$

ko‘rinishga ega. Agar $x(t)$ ekstremal (8) tenglamani qanoatlantirsa, u holda bo‘laklab integrallash bilan

$$\int_0^{2\pi} x\left(x'' + \lambda p(t)x\right)dt = xx' \Big|_0^{2\pi} - \int_0^{2\pi} x'^2(t)dt + \lambda \int_0^{2\pi} p(t)x^2 = 0$$

tenglikka va bundan

$$\int_0^{2\pi} x'^2(t)dt = \lambda$$

tenglikka kelamiz. Demak, topilishi talab qilingan minimal qiymat

$$x'' + \lambda p(t)x = 0,\ x(0) = x(2\pi)$$

Shturm-Liuvil masalasining eng kichik xos qiymati λ dan iborat ekan.

3-usul.

$$z(t) = x(t+\pi) - x(t)$$

Funksiya $t = 0$ va $t = \pi$ nuqtalarda turli ishorali qiymatlar qabul qilgani uchun $(0, \pi)$ oraliqning qandaydir $t = \alpha$ nuqtasida nolga teng bo‘ladi.

Aytaylik, $z(\alpha) = 0, 0 \le \alpha < \pi$ va $x(\alpha) = a$ bo‘lsin.

$x' \in L^2$ bo‘lgani uchun, $(x-a)^2 ctg(t-\alpha)$ funksiya $t = \alpha$ va $t = \alpha + \pi$ nuqtalarda nolga teng bo‘ladi.

Shuning uchun

$$\int_0^{2\pi} \left[x'^2 - (x-a)^2 - \left\{x' - (x-a)ctg(t-\alpha)\right\}^2\right]dt =$$

$$= \left[(x-a)^2 ctg(t-\alpha)\right]_0^{2\pi} = 0.$$

Endi (1) munosabatni inobatga olgan holda quyidagi

$$\int_0^{2\pi} (x'^2 - x^2)dt = 2\pi a^2 + \int_0^{2\pi} \left[x' - (x-a)ctg(t-\alpha)\right]^2 dt$$

tenglikdan

$$\int_0^{2\pi} x'^2 dt \geq \int_0^{2\pi} x^2 dt$$

tengsizlikni hosil qilamiz. Bu tengsizlikda tenglik belgisi $a = 0$ va $x' = xctg(t - \infty)$, $x(t) = C\sin(t - \infty)$ funksiya uchun o‘rinli ekani ravshan.

4-usul. Endi (2) tengsizlikning isbotini L.S.Pontryagin [3] maksimum prinsipiga asoslangan holda olib boramiz. Buning uchun quyidagi optimallashtirish masalasini qaraymiz:

$$\dot{x} = u, \qquad x(0) = x(2\pi), \qquad \int_0^{2\pi} x(t)\,dt = 0$$

Bu shartlar bajarilganda,

$$\int_0^{2\pi} u^2(t)\,dt \geq \int_0^{2\pi} x^2(t)\,dt$$

tengsizlik o‘rinli bo‘lishini ko‘rsatish uchun

$$\int_0^{2\pi} \left(u^2 - x^2 - \lambda x\right) dt \to inf$$

funksionalni qaraymiz.

Pontryagin maksimum prinsipiga ko‘ra

$$H = \psi u - \left(u^2 - x^2 - \lambda x\right)$$

funksiyani tuzamiz. Exstremumning zaruriy shartiga ko‘ra

$$\frac{\partial H}{\partial u} = 0 \Rightarrow \quad u = \frac{\psi}{2}$$

qo‘shma sistema esa

$$\dot{\psi} = -\frac{\partial H}{\partial x} \Rightarrow \dot{\psi} = -2x - \lambda$$

ko‘rinishga ega bo‘lgani uchun

$$\ddot{x} = \dot{u} = \frac{\dot{\psi}}{2} = -\frac{2x + \lambda}{2} = -x - \frac{\lambda}{2},$$

yoki

$$\ddot{x} + x = -\frac{\lambda}{2}$$

o‘zgarmasni variatsiyalash usulini qo‘llab, bu tenglamaning umumiy yechimi

$$x(t) = A\cos t + B\sin t - \frac{\lambda}{2}$$

ko‘rinishda ekanligini aniqlaymiz. Ammo

$$0 = \int_0^{2\pi} x(t)dt = C_{1_0} \int_0^{2\pi} \cos t + C_{20} \int_0^{2\pi} \sin t = -\frac{\lambda}{2} \cdot 2\pi = -\lambda$$

bundan $\lambda = 0$ va shu bilan birga doimo $\int_0^{2\pi} \dot{x}^2(t)\,dt \geq 0$ ekanligi e’tiborga olinsa,

$$\int_0^{2\pi} \left(u^2 - x^2 - 0 \cdot x\right) dt \geq 0 \Rightarrow \int_0^{2\pi} \dot{x}^2 dt \geq \int_0^{2\pi} x^2 dt$$

bu tengsizlikda tenglik belgisi $x(t) = A\cos t + B\sin t$ funksiya uchun bajariladi.

5-usul. Wirtinger tengsizligini variatsion hisob usuli bilan ham keltirib chiqarish mumkin. Buning uchiun

$$\frac{\int_0^{\pi} x'^2(t)\,dt}{\int_0^{\pi} x^2(t)\,dt} \to \min, \quad \int_0^{\pi} x(t)\,dt = 0$$

variatsion hisob masalasini qaraymiz. Bu masalaning yechimi mavjud bo‘lsin deb faraz qilamiz, ya’ni

$$\frac{\int_0^{\pi} x'^2(t)\,dt}{\int_0^{\pi} x^2(t)\,dt} = \lambda$$ bo‘lsin. U holda $$\int_0^{\pi} \left[x'^2(t) - \lambda x^2(t) + x(t) \right] dt \to \min$$

Endi $F = x'^2 - \lambda x^2 + x$ funksiya uchun $F_{x'} = 2x', F_x = -2\lambda x + 1$ bo‘lgani uchun variatsion hisobning $-\frac{d}{dt}F_{x'} + F_x = 0$ Eyler tenglamasiga asosan $x'' + \lambda x = 1$ tenglamaga ega bo‘lamiz. Bu tenglamaning umumiy yechimi

$$x(t) = C_1 \cos\sqrt{\lambda}t + C_2 \sin\sqrt{\lambda}t - \frac{1}{\lambda}$$

bo‘lgani uchun kelgusi hisoblashlar 4-usuldagidek olib boriladi.

Wirtinger tengsizligining izoperimetrik masalaga tatbiqi

(2) – Virtinger tengsizligi doiraning izoperimetrik xossasini isbotlashga asos bo‘ladi.

S yuzali chegaralovchi *L* uzunlikdagi sodda yopiq l chiziq qutb koordinatalari sistemasida

$$l : x = x(\varphi), y = y(\varphi) \quad 0 \le \varphi \le 2\pi$$

tenglama bilan berilgan bo‘lsin. l chiziq yoyining uzunligi bo‘yicha

$$\varphi = \frac{2\pi s}{L}$$

parametr kiritamiz. Qulaylik uchun $x(\varphi)$ va $y(\varphi)$ funksiyalarni differensiallanuvchi funksiyalar deb qaraymiz. l yopiq chiziq bo‘lgani uchun

$$\int_0^{2\pi} y\,d\varphi = 0$$

U holda

$$\left(\frac{dx}{d\varphi}\right)^2 + \left(\frac{dy}{d\varphi}\right)^2 = \left\{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right\}\frac{L^2}{4\pi^2} = \frac{L^2}{4\pi^2},$$ chunki $$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$$

$$\frac{L^2}{2\pi} - 2S = \int_0^{2\pi}\left\{\left(\frac{dx}{d\varphi}\right)^2 + \left(\frac{dy}{d\varphi}\right)^2\right\}d\varphi + 2\int_0^{2\pi} y\frac{dx}{d\varphi}d\varphi =$$

$$= \int_0^{2\pi}\left(\frac{dx}{d\varphi} + y\right)^2 d\varphi + \int_0^{2\pi}\left\{\left(\frac{dy}{d\varphi}\right)^2 - y^2\right\}d\varphi \ge$$

$$\ge \int_0^{2\pi}\left\{\left(\frac{dy}{d\varphi}\right)^2 - y^2\right\}d\varphi \ge 0.$$

Shunig uchun

$$S \le \frac{L^2}{4\pi}$$

izoperimetrik tengsizlik o‘rinli.

Bu yerda tenglik $y = A\cos\varphi + B\sin\varphi$ va $x = -\int y\,d\varphi = -A\sin\varphi + B\cos\varphi + C$ bo‘lganda, ya’ni $l : (x - C)^2 + y^2 = A^2 + B^2$ doira uchun bajariladi.

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